Braneworld Inflation

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Submitted for the degree of M.Sc. in Cosmology

27th August 2003

DECLARATION

I have read and understood the definition of ‘Plagiarism’ as set out in the **General Assessment Handbook for Masters and Postgraduate Diploma and Postgraduate Certificate Candidates** under item 12.

This dissertation is my own work, except where explicitly stated.

Some of the text and figures of Chapters 4 and 5 are taken from my contribution to Liddle and Smith (2003), which has been accepted for publication in Physical Review D.

Signed . . . . Date . . . .

Anthony Smith

Abstract

Chaotic inflation on the Randall–Sundrum type II braneworld is investigated. After introducing inflation and the braneworld, I derive predictions for perturbations arising from monomial potentials, and compare these with observational constraints from the cosmic microwave background and large-scale structure, including WMAP and the 2dF galaxy power spectrum. I find results for any monomial potential in the low-energy (standard cosmology) and high-energy limits, and results for the quadratic and quartic potentials for any inflationary energy scale. Potentials with exponent greater than four are strongly disfavoured by the data. For the quadratic potential, the perturbations generated by inflation taking place entirely in the high-energy regime are further from scale-invariance than in the low-energy limit; with a quartic potential there is little difference; and for potentials with higher exponent the trend is reversed, with high energies driving the perturbations nearer to scale-invariance. In the intermediate regime, where the inflationary energy scale is comparable to the brane tension, there are greater deviations from scale-invariance for the quartic and quadratic potential than in either the high- or low-energy limits, with the effect particularly strong for the quartic potential. Constraints on the exponent of monomial potentials are found to be stronger in the braneworld scenario than in the standard inflationary cosmology.

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Chapter 1

Introduction

The Universe has become much less simple within the last generation. For example, string theory and M-theory have suggested that space has several additional dimensions. Usually these extra dimensions are considered to be very small and effectively negligible, but it has recently been shown that they need not be so small. A simple model which utilises this idea is the **Randall-Sundrum type II braneworld**. This has one extra large dimension, leading to a spacetime with four large spatial dimensions and one time dimension. Our observable Universe is confined to the **brane**, which exists in the higher-dimensional **bulk** spacetime.

In today's Universe, this braneworld scenario would be effectively indistinguishable from the ordinary four-dimensional Universe (three spatial dimensions, and one time dimension). However, higher energies would change the properties of gravity which we observe. Such high energies are thought to have filled the very early Universe, according to the standard Big Bang cosmology. This motivates investigations of **inflation** on the brane. Inflation is an early Universe phenomenon, which is thought to have left signatures in the cosmic microwave background (CMB). It is hoped that observations (for example, of the CMB) may make it possible to determine whether the Randall–Sundrum braneworld is a good description of the Universe in which we live. This is the motivation behind my thesis.

In Chapter 2, the standard **Big Bang** cosmology is introduced, along with the concept of inflation. Expressions are derived for certain properties of the perturbations generated by **chaotic inflation**, and a connection is made with the anisotropies observed in the cosmic microwave background.

Chapter 3 introduces the braneworld scenario, showing how the cosmological equations are modified. The effect on inflation is described, and modified expressions for the perturbations are derived.

In Chapter 4, I derive detailed predictions for a number of different potentials for the scalar field which drives inflation. The potentials considered are monomial. I find results for any monomial potential in the high- and low-energy limits, and results for the quadratic and quartic potentials for any energies. The predictions are produced in a form which enables comparison with observation.

This comparison is made in Chapter 5, where the observational constraints come from recent observations of the CMB (including WMAP) and large-scale structure. I compare my predictions with these observations, and find that monomial potentials come under stronger observational pressure in the braneworld scenario than in the standard cosmology.

The implications of these results and prospects for further investigation are discussed in the Conclusions (Chapter 6)—in particular, how it might be possible to evaluate the Randall–Sundrum braneworld scenario by observations.

Chapter 2

Standard Inflation

2.1 Cosmology in the ordinary

2.1.1 Cosmological principle

Liddle and Lyth write,

> The central premise of modern cosmology is that, at least on large scales, the Universe is homogeneous and isotropic. (Liddle and Lyth, 2000, p. 12)

This premise is known as the **cosmological principle**, or **Copernican principle**, after Copernicus, who popularised the belief that the Sun, rather than the Earth, was at the centre of the Solar System. The cosmological principle takes the ideas of Copernicus to a new level, stating that we not only occupy no special position in the Solar System, but that we are also nowhere special in the Universe as a whole. Roos writes,

> The history of ideas on the structure and origin of the Universe shows that mankind has always put itself at the centre of Creation. As astronomical evidence has accumulated, these anthropocentric convictions have had to be abandoned one by one. From the natural idea that the solid Earth is at rest and the celestial objects all rotate around us, we have come to understand that we inhabit an average-sized planet orbiting an average-sized sun, that the solar system is in the periphery of a rotating galaxy of average size, flying at hundreds of kilometres per second towards an unknown goal in an immense Universe, containing billions of similar galaxies. (Roos, 1994, p. 1)

We can only observe the Universe from one location, and from the Earth the Universe appears isotropic on large scales. This is most noticeable in the **cosmic microwave background** (CMB), which is isotropic to one part in a hundred thousand, after subtracting our motion with respect to the CMB. But why should we then think that the Universe would appear isotropic to all observers, irrespective of their location? Could not isotropy at our location merely imply that we are near the centre of a spherical distribution of matter? In his best-selling book, *A Brief History of Time*, Stephen Hawking addresses this question:

> Now at first sight, all this evidence that the universe looks the same whichever direction we look in might seem to suggest there is something special about our place in the universe. In particular, it might seem that if we observe all other galaxies to be moving away from us, then we must be at the center of the universe. There is, however, an alternate explanation: the universe might look the same in every direction as seen from any other galaxy, too. This, as we have seen, was Friedmann's second assumption. We have no scientific evidence for, or against, this assumption. We believe it only on grounds of modesty: it would be most remarkable if the universe looked the same in every direction around us, but not around other points in the universe! (Hawking, 1988, p. 42)

Thus the basis for the cosmological principle—that we are nowhere special in the Universe—is an unprovable assumption. This, Hawking claims, is motivated by modesty, but it is more fundamentally motivated by one's religious beliefs. It follows from the belief that life is a purposeless accident, and that human beings have no special significance in the Universe. The foundation of the cosmological principle, and hence of modern cosmology, is no less than an assumption of atheistic naturalism.

Of course, if our existence were purposed by a divine being, as a large proportion of the Earth's intelligent inhabitants would claim, it should be no surprise if we *do* occupy a privileged position in the Universe—that is, if our Galaxy were near the centre. This alternative assumption (that the Universe is bounded, in contrast to the cosmological principle's unbounded cosmos) leads to a **galactocentric** cosmology, which would be very different from the standard cosmology. Some galactocentric models exist, for example, one which has the Universe emerging from within a white hole (Humphreys, 1994), but they are at an early stage of development.

It is true that the standard cosmology has produced a number of impressive predictions, such as the existence and structure of the CMB and the abundances of light elements. These lend considerable support to the validity of the cosmological principle. Nonetheless, they do not raise the status of this principle any higher than 'plausible assumption', nor do they negate the legitimacy of cosmologies founded on different principles. It is only *after* these alternative models have been seriously studied that a fair comparison may be made.

In the remainder of this thesis, however, the cosmological principle will be assumed, along with the standard **Big Bang** cosmology which follows from it. But it is worth emphasizing the atheistic assumptions undergirding this model, and the fact that theists (such as myself) have no reason to make these assumptions.

2.1.2 Relativistic Cosmology

Before Einstein, the accepted dogma about the Universe was that it was infinite and static. Newton (1642–1727) understood the stars to be suns like our own, evenly distributed throughout infinite space. He reasoned (incorrectly) that such a distribution of matter would be stable, allowing a static Universe. In fact, such a Universe would collapse. Leibniz (1646–1716), a contemporary of Newton, also rejected the idea of a finite Universe, because he believed that a finite Universe would necessarily be bounded, and thus have a centre.

With Riemann (1826–1866) came the realization that a finite Universe need not be bounded. This follows from his work on non-Euclidean geometry, particularly on curved space. Riemann investigated geometries in which Euclid's *parallel postulate* did not hold. With this freedom, it is possible to have geometries which are both finite and unbounded—analogous to a sphere in two dimensions—so that travelling in a straight line will eventually bring you back to your starting point.

Einstein (1879–1955) published his special theory of relativity in 1905, which dealt with observers moving relative to each other in inertial frames. His monumental work on gravity, the general theory of relativity, was announced in 1915, making full use of Riemann's ideas about geometry. In general relativity, acceleration is indistinguishable from being in a gravitational potential, and both are related to the curvature of space and time. General relativity has formed the basis of all major work in cosmology in subsequent years.

Initially Einstein was disturbed by the results of his theory, since they suggested that the Universe could not be static. He therefore added the **cosmological term** to his equations, the coefficient of which was a valid constant of integration, known as the **cosmological constant**. This acted like a repulsive force, which could be fine-tuned to exactly balance the attractive force due to the matter in the Universe. However, as was realized some years later, this configuration was unstable. If there was slightly more matter in the Universe than the cosmological constant could hold in place, the Universe would proceed to collapse; if there was too little matter, the Universe would expand without limit. Einstein was later to describe the cosmological constant as his greatest blunder but, ironically, within the last few years, most cosmologists have become convinced that there is a non-zero cosmological constant, or something else which produces the same effect (e.g. *quintessence*).

It was not until after Hubble's (1889–1953) observations of galactic redshifts in 1925 that the idea of a non-static Universe was given serious consideration. In the spectra of nearby galaxies, Hubble observed a trend for galaxies to be moving away from us. The light was, in general, shifted towards the red end of the spectrum, interpreted as a Doppler shift from a receding galaxy. The trend was clear, and has been confirmed by observations of a vast number of galaxies since then. This was taken as compelling evidence for the expansion of the Universe, and Einstein was finally convinced of this in 1929.

The cosmological principle, plus general relativity, has lead to the standard model of cosmological expansion, commonly referred to by some combination of the names Friedmann, Robertson, Walker and Lemaître.

Einstein's equations of general relativity are written as the tensor equation,

$$G_{\mu\nu} = 8\pi G\, T_{\mu\nu} + \Lambda g_{\mu\nu}\,, \tag{2.1}$$

where the signs of terms may vary according to differing conventions. Here, the tensor $g_{\mu\nu}$ is the metric, $G_{\mu\nu}$ is the Einstein tensor (a function of the metric), and $T_{\mu\nu}$ is the energy-momentum tensor, describing the distribution and properties of the matter and energy. Λ is the famous cosmological constant, and G is Newton's constant of gravitation. In this equation, and in the rest of the thesis, units have been assumed in which the speed of light, c, is equal to 1. It will also be assumed that Planck's constant, $\hbar$, is equal to 1.

For a homogeneous and isotropic spacetime, the metric $g_{\mu\nu}$ must be the Robertson–Walker metric. In a certain coordinate system, this leads to the line element,

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \mathrm{d}t^2 - a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right]. \tag{2.2}$$

(In this form, it has *not* been assumed that k, the constant describing the curvature, may only be equal to 1, 0 or -1.) a is the **scale factor**, which describes the size of the Universe. Distances between two points in a homogeneous and isotropic Universe are expressed as the product of the scale factor (which is a function of time) and the comoving distance (which is constant in time).

The matter/energy content of the Universe is usually assumed to be a *perfect fluid* with energy density ρ and pressure p. In the same choice of coordinates used above, the energy-momentum tensor for this perfect fluid is

$$T_{\mu\nu} = \mathrm{diag}(\rho, -p, -p, -p)\,. \tag{2.3}$$

The Einstein equations, (2.1), with the Robertson–Walker metric and this energy-momentum tensor, lead to the following equations. The **Friedmann equation** describes the rate of expansion:

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3}\,, \tag{2.4}$$

where H is the **Hubble parameter**. The **acceleration equation** tells us whether the expansion will accelerate or decelerate:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3} \tag{2.5}$$

The Friedmann and acceleration equations combine to give the **fluid equation** (or continuity equation), which represents conservation of energy:

$$\dot{\rho} + 3H(\rho + p) = 0\,. \tag{2.6}$$

If we know the density and pressure of the constituents of the (homogeneous) Universe, these equations enable us to precisely describe how the Universe evolves.

2.1.3 The Big Bang

An expanding Universe, when taken back in time, would have been much more dense. What happens when we go back in time as far as possible, to when today's observable Universe was infinitely dense and infinitesimally small? This 'singularity' is known as the **Big Bang**. The 'bang' itself is somewhat blurry, since no one really knows what the properties of such a high-density Universe would be. This is because before the *Planck time*, general relativity must be combined with quantum mechanics—something which has not yet been achieved. However, the time at which the 'bang' would have happened is labelled as the time $t = 0$.

The situation at very early times (before, say, $t = 10^{-32}\,\mathrm{s}$) is even more blurry when one considers inflation, as we will shortly. In one model of inflation, *chaotic inflation*, many Universes could have existed for vast eons of time *before* $t = 0$!

However, after inflation the standard cosmological model is much clearer, and will be outlined in this section.

Whether or not inflation happens, at $t = 10^{-2}\,\mathrm{s}$ we find that the Universe would have been dominated by relativistic particles ('radiation domination'), and would have had a temperature of roughly 10^{11} K. There would have been photons, neutrinos, electrons, positrons, protons and neutrons, all in thermal equilibrium. At $t \simeq 100\,\mathrm{s}$, **nucleosynthesis** takes place. This is when the temperature has dropped sufficiently (to roughly 10^9 K) to allow protons and neutrons to combine to form atomic nuclei, predominantly hydrogen and helium-4. The density of radiation falls more quickly than that of matter, so there comes a point when the matter becomes dominant. This would have happened at $t \simeq 10^4$ yr. At this time, the energy of the photons is still sufficiently high to prevent electrons from combining with atomic nuclei to form atoms. When the temperature had dropped to around 3000 K (after about 350 000 years), this would no longer have been the case, and atoms were formed. This is the epoch of **decoupling**, and **recombination**, and corresponds to the formation of the **cosmic microwave background** (see Section 2.2.4).

After this point, the Universe is essentially in the same physical state as it is now. Any inhomogeneities in the matter would grow, forming stars and galaxies, although this process is not well understood at present.

The problem at this stage is: where did the inhomogeneities come from? Why is the Universe not perfectly smooth? This question was eventually answered with the idea of inflation.

2.2 Inflation in the ordinary

2.2.1 Motivation

Before the 1980s it was realized that there were certain problems with the standard Big Bang cosmology. It appeared that the initial conditions[1] were somewhat surprising. These problems were resolved with the concept of **inflation**. However, the main attraction of inflation is that it provides a mechanism for generating **structure**.

Inflation is the name given to a very short period in the early Universe during which the expansion was accelerating. This corresponds to $\ddot{a} > 0$, or, equivalently, $\mathrm{d}(aH)/\mathrm{d}t > 0$. Hence the **comoving Hubble length**, H^{-1}/a, is decreasing during inflation.

An inflationary model was first proposed by Starobinsky (1979), but this was a complicated model, and did not spread outside the Soviet Union at that time. Inflation was properly invented in 1981, when Alan Guth independently published a paper, and coined the term 'inflation' (Guth, 1981). In that paper, he proposed inflation as a solution to the problems with the standard Big Bang cosmology, which will be discussed briefly below. These were primarily the horizon and flatness problems, but he also mentioned the monopole problem.

The **horizon problem** is based on the observed isotropy of the cosmic microwave background (see Section 2.2.4), which exhibits a black-body spectrum from all directions characteristic of precisely the same temperature, 2.73 K, to one part in a hundred thousand. This implies that these points in the sky were once in thermal equilibrium. But how can this be, if they have not had time to communicate since the Big Bang? Two points can only be in thermal equilibrium if the distance between them is less than the *particle horizon*, roughly given by cH^{-1}, where c is the speed of light, and H^{-1} is a rough estimate of the age of the Universe. During inflation, H is almost constant, so inflation rapidly increases the size of the Universe while keeping the horizon scale roughly constant. (This is equivalent to the comoving Hubble length decreasing.) So the whole of today's observable Universe could have been within the horizon before inflation started. This is illustrated in Fig. 2.1.

The **flatness problem** concerns the observed flatness of the Universe, most noticeable in the cosmic microwave background. A flat Universe corresponds to $k = 0$ in the Friedmann equation (2.4). This is equivalent to $\Omega_{\rm tot} \equiv \Omega + \Omega_\Lambda = 1$ with the density parameters defined as $\Omega_{\rm X} \equiv \rho_{\rm X}/\rho_{\rm c}$, where

$$\rho_{\rm c} \equiv \frac{3H^2}{8\pi G}\,, \tag{2.7}$$

is the **critical density**. The density parameter for the cosmological constant is defined as $\Omega_\Lambda = \Lambda/3H^2$. In Friedmann's models, where the cosmological constant is zero, a Universe of critical density is the borderline between a closed Universe, which will collapse, and an open Universe,

[1] *Initial conditions* here refers to the conditions at some early time when the energy scale was comfortably below the Planck scale. Conditions at energies higher than this are not accessible using classical general relativity, since quantum effects are thought to dominate.

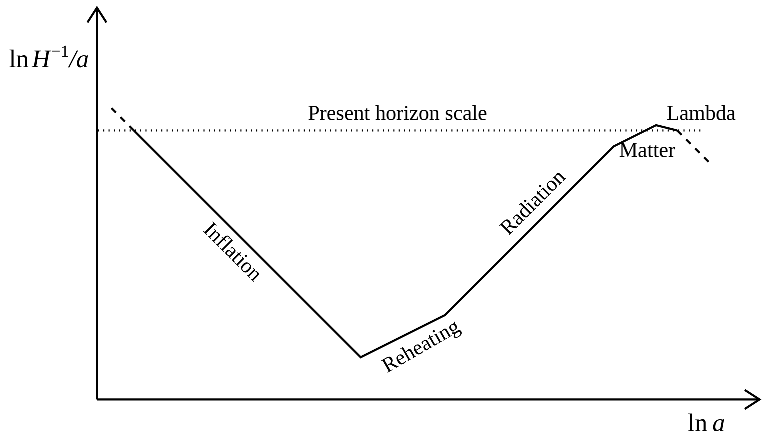

Figure 2.1: The comoving Hubble length against the scale factor (from Liddle and Leach, 2003). Different epochs in the history of the Universe are labelled, finishing with domination by radiation, matter and then the cosmological constant, Lambda. It can be seen that the present horizon scale was also the horizon scale at some time during inflation. This explains the *horizon problem*, regarding points separated today by more than the horizon having been in causal contact in the early Universe.

which will expand forever. With a positive cosmological constant, however, it is possible to have a closed Universe which will expand forever.

From Eq. (2.4), we obtain,

$$\Omega_{\rm tot} - 1 = \Omega + \Omega_\Lambda - 1 = \frac{k}{a^2 H^2}\,. \tag{2.8}$$

Now, in a Universe dominated by matter or radiation, the expansion will always be decelerating. So $\dot{a}^2 = a^2 H^2$ will decrease with time. Hence, unless $\Omega_{\rm tot}$ is precisely equal to 1 for all time, it will always be moving away from 1. A flat Universe is therefore unstable, and it begs some explanation as to why it was flat initially.

Inflation solves the flatness problem because, during inflation, aH is increasing, so the Universe is driven towards a flat state. The Universe can be made arbitrarily flat by invoking a suitably lengthy period of inflation. This is analogous to a balloon, which, though not flat, may be inflated so that a particular area of its surface appears flat.

The **monopole problem** concerns relic particle abundances predicted by **grand unified theories** (GUTs) of particle physics. These imply the existence in the very early Universe of certain unusual particles, most famously of magnetic monopoles (but also of particles such as gravitinos and moduli fields) which should exist today. These particles would have been non-relativistic for long enough to enable them to become dominant over radiation in the Universe. However, such particles have not been found. This fact may be explained by inflation, which would have diluted the density of such particles to make them so rare that we would not expect to find them.

As well as making the Universe highly flat and homogeneous, inflation also introduces inhomogeneities. This is because the microscopic quantum fluctuations, which exist everywhere, are rapidly expanded to macroscopic sizes, and 'frozen in'. It is these fluctuations which form the seeds of structure formation in the subsequent evolution of the Universe.

Guth's model became known as **old inflation**. There were various problems with it, for example, that it produced too much inhomogeneity, and it was soon abandoned. **New inflation** was proposed independently by Albrecht and Steinhardt (1982) and by Linde (1982), but suffered a similar fate due to various problems, for example, that it requires an extremely flat potential for the scalar field (see below). Today there is a huge number of models of inflation. One popular model, which has lasted for over 20 years and which will be the focus of this thesis, is **chaotic inflation**, also proposed by Linde (1983).

GUTs predict the existence of **scalar fields**, associated with the splitting (symmetry breaking) of the fundamental forces (gravity, the strong force, the weak force, and the electromagnetic force), which would have been unified as one force before the Planck time (10^{-43} seconds). Inflation is thought to have been driven by such scalar field, commonly called the **inflaton**, which may be a result of symmetry breaking, but it may be an independent field not directly related to symmetry breaking. In chaotic inflation, the scalar field may take different forms and values at different points, due to quantum fluctuations. Only those regions which have a suitable scalar field can undergo inflation. This leads to the idea of a **multiverse**, where our observable Universe is just one of perhaps infinitely many disconnected regions which have undergone inflation.

In fact, this can lead to an unending 'self-reproducing inflationary universe' (Linde, 1994), a concept known as **eternal inflation**. In this scenario, the whole universe (or multiverse) develops as a complicated fractal-like structure, with inflationary 'bubbles' emerging inside regions which have already inflated.

2.2.2 Dynamics of inflation

From the Friedmann equation (2.4), the rate of expansion is approximately proportional to the square root of the energy density (neglecting the cosmological constant). As will be shown below, the energy density of a scalar field decreases much more slowly than the energy density of normal matter or radiation, and may remain almost constant, leading to exponential expansion (since constant H^2 implies $a \propto e^{Ht}$), as found in a Universe dominated by a cosmological constant. This is what happens during inflation, when the scalar field acts like an effective cosmological constant.

The value of the scalar field changes until it reaches a value corresponding to a minimum of the potential energy of the field. (In the simplest models, this minimum corresponds to the scalar field vanishing.) The scalar field is said to 'roll down' the potential. Fig. 2.2 shows an example of a simple potential for a scalar field. Once the scalar field nears the minimum of the potential, it begins to oscillate about this point, causing a period of **reheating**.

A real scalar field, ϕ, is described by its Lagrangian density,

$$\mathcal{L} = \frac{1}{2}\partial^\mu \phi \, \partial_\mu \phi - V(\phi) \, , \tag{2.9}$$

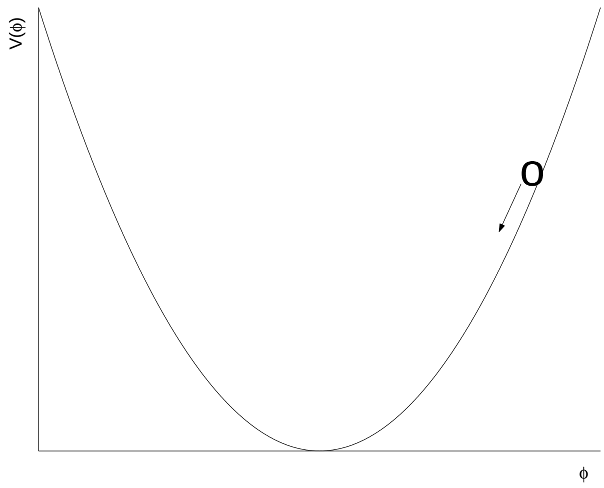

Figure 2.2: An example of a scalar field, with a quadratic potential, $V \propto \phi^2$. The scalar field 'rolls down' the potential towards the minimum.

where V is the potential. Taking the field to be homogeneous, this is used to obtain the energy-momentum tensor, from which the density and pressure of the scalar field (respectively) may be found (Liddle and Lyth, 2000, p. 332):

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi)\,, \tag{2.10}$$

$$p = \frac{1}{2}\dot{\phi}^2 - V(\phi)\,, \tag{2.11}$$

During inflation, the curvature term of the Friedmann equation (2.4) rapidly becomes negligible, and we neglect the cosmological constant. With ρ for the scalar field, this gives,

$$H^2 = \frac{8\pi}{3M_{\mathrm{P}}^2}\left(\frac{1}{2}\dot{\phi}^2 + V\right)\,, \tag{2.12}$$

where $M_{\mathrm{P}} = G^{-1/2}$ is the Planck mass.

From the fluid equation (2.6),

$$\dot{\rho} = -3H(\rho + p)\,, \tag{2.13}$$

we obtain,

$$\ddot{\phi}\dot{\phi} + \frac{\mathrm{d}V}{\mathrm{d}\phi}\dot{\phi} = -3H\dot{\phi}^2\,, \tag{2.14}$$

$$\ddot{\phi} + V' = -3H\dot{\phi}\,, \tag{2.15}$$

where the prime denotes differentiation with respect to ϕ.

Inflation takes place when $\ddot{a} > 0$. So (neglecting the cosmological constant) the acceleration equation (2.5) implies that,

$$p < -\frac{\rho}{3}\,, \tag{2.16}$$

during inflation. With a scalar field, this is equivalent to

$$\dot{\phi}^2 < V \,. \tag{2.17}$$

If we impose slightly stronger conditions on the scalar field, the equations become much easier to solve. (2.17) says that the field varies more slowly than a certain limit. We therefore make approximations on the scalar field, stating that it varies slowly. These are the **slow-roll conditions**:

$$\frac{1}{2}\dot{\phi}^2 \ll V \,, \tag{2.18}$$

$$|\ddot{\phi}| \ll |3H\dot{\phi}| \,. \tag{2.19}$$

We can now see why a slowly-rolling scalar field gives the expansion rate referred to above. The fluid equation, with $\dot{\phi}$ being small, implies that $\dot{\rho}$ is also small, i.e., that the energy density of the scalar field is roughly constant. This gives a near-exponential rate of expansion.

With the slow-roll approximation, Eqs. (2.12) and (2.15) give us the two equations to describe the dynamics of the inflationary expansion,

$$H^2 \simeq \frac{8\pi}{3M_{\mathrm{P}}^2} V \,, \tag{2.20}$$

$$3H\dot{\phi} \simeq -V' \,, \tag{2.21}$$

where an approximate equality sign denotes equality under the slow-roll approximation.

The slow-roll conditions, (2.18) and (2.19), are used to place constraints on the **slow-roll parameters**, defined as follows:

$$\epsilon \equiv \frac{M_{\mathrm{P}}^2}{16\pi}\left(\frac{V'}{V}\right)^2 \,, \tag{2.22}$$

$$\eta \equiv \frac{M_{\mathrm{P}}^2}{8\pi}\left(\frac{V''}{V}\right) \,. \tag{2.23}$$

During slow-roll inflation, (2.18) implies that $\epsilon \ll 1$ and (2.19) implies that $|\eta| \ll 1$.

We also note that Eqs. (2.20) and (2.21) imply

$$\epsilon \simeq -\frac{\dot{H}}{H^2} \,, \tag{2.24}$$

and that

$$\frac{\dot{H}}{H^2} = \frac{\ddot{a}}{a}\frac{1}{H^2} - 1 \,. \tag{2.25}$$

Hence inflation takes place by definition when

$$\frac{\dot{H}}{H^2} > -1 \,, \tag{2.26}$$

i.e., when $\epsilon \lesssim 1$. This is why the end of inflation is often defined to be $\epsilon = 1$, for the purpose of predicting perturbations, but the definition is not exact.

The amount of inflation is quantified by the number of ***e*-folds**, N, where an e-fold is an increase in the size of the the Universe by a factor of e. Thus,

$$
\begin{aligned}
N &= \ln \frac{a_{\rm f}}{a_{\rm i}} && (2.27)\\
&= \int_{a_{\rm i}}^{a_{\rm f}} \frac{{\rm d}a}{a} && (2.28)\\
&= \int_{t_{\rm i}}^{t_{\rm f}} H\,{\rm d}t && (2.29)\\
&= \int_{\phi_{\rm i}}^{\phi_{\rm f}} \frac{3H^2}{3H\dot{\phi}}\,{\rm d}\phi && (2.30)\\
&\simeq -\frac{8\pi}{M_{\rm P}^2}\int_{\phi_{\rm i}}^{\phi_{\rm f}} \frac{V}{V'}\,{\rm d}\phi\,, && (2.31)
\end{aligned}
$$

from Eqs. (2.20) and (2.21). The subscripts ‘i’ and ‘f’ denote the initial and final values of the relevant quantities.

At the end of inflation, various events take place, which recover the conditions in the Universe at this stage of the standard Hot Big Bang expansion. This is the period of **reheating**, the usual model of which has the energy density of the inflaton field being converted into matter and radiation. There are alternative models of reheating, one of which is *curvaton reheating*, which has recently been proposed, and will be mentioned briefly in Chapter. 5. A detailed account of reheating will not be presented here, as it does not directly affect the production of density perturbations.

2.2.3 Growth of structure

As well as generating large-scale homogeneity in the Universe, inflation also fills it with inhomogeneities. These seeds of structure are formed from random quantum fluctuations in the scalar field. The perturbations generated are **Gaussian** and **adiabatic**. In this section, we follow the definitions and derivations of Liddle and Lyth (2000).

In the inflating region, the scalar field ϕ, is taken to consist of a homogeneous component, $\phi_0(t)$ and a small perturbation, $\delta\phi(\mathbf{x}, t)$. The resulting (inhomogeneous) scalar field, $\phi = \phi_0 + \delta\phi$, will obey

$$\ddot{\phi} + 3H\dot{\phi} - \nabla^2\phi + V' = 0\,, \tag{2.32}$$

where $\nabla^2 = a^{-2}\sum \partial^2/\partial x_i^2$. Using a Taylor expansion, $V'(\phi_0 + \delta\phi) \simeq V'(\phi_0) + \delta\phi\, V''(\phi_0)$, and recalling Eq. (2.15) for the homogeneous part of the field, we obtain,

$$\ddot{\delta\phi} + 3H\dot{\delta\phi} - \nabla^2\delta\phi + V''(\phi_0)\delta\phi = 0\,. \tag{2.33}$$

With a Fourier expansion,

$$\delta\phi(\mathbf{x}, t) = \sum_{\mathbf{k}} \delta\phi_{\mathbf{k}}(t)e^{\mathbf{k}.\mathbf{x}}\,, \tag{2.34}$$

each Fourier component therefore satisfies,

$$\delta\ddot{\phi}_{\mathbf{k}} + 3H\delta\dot{\phi}_{\mathbf{k}} + \left(\frac{k}{a}\right)^2 \delta\phi_{\mathbf{k}} + V''(\phi_0)\delta\phi_{\mathbf{k}} = 0\,. \tag{2.35}$$

A given fluctuation will be 'frozen in' when its size exceeds the horizon scale. This occurs when $k = aH$, and the moment at which the fluctuation is frozen in is called **horizon crossing** or **horizon exit**.

If the horizon crossing takes place during slow-roll inflation, the condition $|\eta| \ll 1$ implies that $V'' \ll H^2$ (cf. Eqs. [2.20] and [2.23]). Hence, until a while after horizon crossing, we may neglect the final term on the left-hand side of Eq. (2.35). (Before horizon crossing, $k > aH$, so the term may be neglected during this period.)

After the fluctuation has left the horizon, the quantity we are most interested in is the **curvature perturbation**, $\mathcal{R}_{\mathbf{k}}$. This is a change in the spatial curvature produced by the perturbation in the scalar field, and is defined, for a flat Universe ($k = 0$) with zero cosmological constant, by

$$H^2(\mathbf{x}, t) \equiv \frac{8\pi G}{3}\rho(\mathbf{x}, t) + \frac{2}{3}\nabla^2\mathcal{R}(\mathbf{x}, t)\,. \tag{2.36}$$

It has the property that it remains almost constant while well outside the horizon. The value it takes during this period is called its **primordial** value. The primordial curvature perturbation satisfies

$$\mathcal{R}_{\mathbf{k}} = -\left[\frac{H}{\dot{\phi}}\,\delta\phi_{\mathbf{k}}\right]_{t=t_*}\,, \tag{2.37}$$

where the time t_* is a few Hubble times after horizon crossing, when the perturbation has settled to a constant value.

The **spectrum** of the curvature perturbation is given by

$$\mathcal{P}_{\mathcal{R}}(k) = \left[\left(\frac{H}{\dot{\phi}}\right)\mathcal{P}_{\phi}(k)\right]^2_{t=t_*}\,, \tag{2.38}$$

where the spectrum for ϕ is given by

$$\mathcal{P}_{\phi}(k, t_*) = \left(\frac{H(t_*)}{2\pi}\right)^2\,. \tag{2.39}$$

During slow-roll inflation, the variation in H, and also in $\dot{\phi}$, is negligible over a few Hubble times, so we may evaluate these quantities at horizon crossing, rather than at $t = t_*$. Hence,

$$\mathcal{P}_{\mathcal{R}}(k) = \left[\left(\frac{H}{\dot{\phi}}\right)\left(\frac{H}{2\pi}\right)\right]^2_{k=aH}\,. \tag{2.40}$$

We define a quantity, A_{S} (also known as δ_{H}, where 'H' stands for 'horizon entry'), which is an approximate measure of the root mean square value of the total density perturbation, $\delta \equiv \delta\rho/\rho$, when it re-enters the horizon. This is related to the spectrum of the curvature perturbation by

$$A_{\mathrm{S}}^2 = \frac{4}{25}\,\mathcal{P}_{\mathcal{R}}\,, \tag{2.41}$$

where the subscript 'S' stands for 'scalar'.

The fluctuations in the inflaton field will also produce **tensor perturbations**, i.e., gravitational wave perturbations. These have spectrum

$$\mathcal{P}_{\rm grav}(k) = \frac{2}{M_{\rm P}^2}\left(\frac{H}{2\pi}\right)^2\Bigg|_{k=aH} . \tag{2.42}$$

In the following, we use a related quantity, $A_{\rm T}$, where the subscript 'T' stands for 'tensor'. This is defined by

$$A_{\rm T}^2 = \frac{8\pi}{25}\,\mathcal{P}_{\rm grav} . \tag{2.43}$$

The two are almost identical, since $8\pi \simeq 25$. (The normalization is essentially arbitrary: this choice corresponds to $A_{\rm T}^2/A_{\rm S}^2 = \epsilon$ in the slow-roll approximation.)

In almost all models of inflation, it is possible to take the spectra to obey simple power laws, with **spectral indices**, n and $n_{\rm T}$, defined by

$$A_{\rm S}^2 \;\propto\; k^{n-1} , \tag{2.44}$$

$$A_{\rm T}^2 \;\propto\; k^{n_T} . \tag{2.45}$$

For the scalar perturbations, the special case $n = 1$ is called the **scale-invariant** or **Harrison–Zel'dovich** spectrum.

It is worth noting that recent observations from the WMAP satellite have suggested that the scalar spectral index may vary, or **run**, from $n > 1$ on large scales to $n < 1$ on short scales (Peiris et al., 2003). This has led to attempts to generate perturbations with this property from inflation, e.g., Chung et al. (2003). But it has also been claimed that the current data are insufficient to measure the running of n (Leach and Liddle, 2003b). In all that follows, the scalar spectral index will be assumed to be constant.

Under the slow-roll approximation, the spectrum of scalar perturbations may be found using Eqs. (2.20) and (2.21) to be (with all expressions evaluated at $k = aH$)

$$A_{\rm S}^2 \;=\; \frac{4}{25}\left[\left(\frac{H}{\dot\phi}\right)\left(\frac{H}{2\pi}\right)\right]^2 \tag{2.46}$$

$$\;=\; \frac{9H^6}{25\pi^2(3H\dot\phi)^2} \tag{2.47}$$

$$\;\simeq\; \frac{9}{25\pi^2}\left(\frac{8\pi}{3M_{\rm P}^2}V\right)^3\frac{1}{(-V')^2} \tag{2.48}$$

$$\;=\; \frac{512\pi}{75M_{\rm P}^6}\,\frac{V^3}{V'^2} , \tag{2.49}$$

and the spectrum of tensor perturbations to be

$$A_{\mathrm{T}}^2 = \frac{4}{25\pi M_{\mathrm{P}}^2} H^2 \tag{2.50}$$

$$\simeq \frac{4}{25\pi M_{\mathrm{P}}^2} \left(\frac{8\pi}{3M_{\mathrm{P}}^2} V \right) \tag{2.51}$$

$$= \frac{32}{75 M_{\mathrm{P}}^4} V \,. \tag{2.52}$$

The ratio between the tensor and scalar perturbations, defined as

$$R \equiv 16 \, \frac{A_{\mathrm{T}}^2}{A_{\mathrm{S}}^2} \,, \tag{2.53}$$

is therefore, under the slow-roll approximation,

$$R \simeq 16 \left(\frac{32}{75 M_{\mathrm{P}}^4} V \right) \left(\frac{75 M_{\mathrm{P}}^6}{512\pi} \frac{V'^2}{V^3} \right) \tag{2.54}$$

$$= 16 \, \frac{M_{\mathrm{P}}^2}{16\pi} \left(\frac{V'}{V} \right)^2 \tag{2.55}$$

$$= 16\epsilon \,, \tag{2.56}$$

where ϵ is one of the slow-roll parameters, defined in Eq. (2.22).

We may also find an expression for $n-1$ under the slow-roll approximation. It is,

$$n - 1 = \frac{\mathrm{d} \ln A_{\mathrm{S}}^2}{\mathrm{d} \ln k} \tag{2.57}$$

$$= \frac{\mathrm{d} \ln A_{\mathrm{S}}^2}{\mathrm{d}\phi} \frac{\mathrm{d}\phi}{\mathrm{d} \ln k} \tag{2.58}$$

$$\simeq \left(\frac{\mathrm{d} \ln V^3}{\mathrm{d}\phi} - \frac{\mathrm{d} \ln V'^2}{\mathrm{d}\phi} \right) \frac{\mathrm{d}\phi}{\mathrm{d} \ln k} \,. \tag{2.59}$$

At $k = aH$, we have $\mathrm{d} \ln k = \mathrm{d} \ln(aH)$. H is almost constant, and, fixing the end of inflation, Eq. (2.27) implies that $\mathrm{d} \ln a_{\mathrm{i}} = -\mathrm{d}N$. Hence,

$$\mathrm{d} \ln k \simeq -\mathrm{d}N \tag{2.60}$$

$$\simeq -\frac{8\pi}{M_{\mathrm{P}}^2} \frac{V}{V'} \, \mathrm{d}\phi \,, \tag{2.61}$$

from Eq. (2.31). Hence,

$$n - 1 \simeq -\left(3 \, \frac{\mathrm{d} \ln V}{\mathrm{d}\phi} - 2 \, \frac{\mathrm{d} \ln V'}{\mathrm{d}\phi} \right) \frac{M_{\mathrm{P}}^2}{8\pi} \frac{V'}{V} \tag{2.62}$$

$$= -\left(3 \, \frac{V'}{V} - 2 \, \frac{V''}{V'} \right) \frac{M_{\mathrm{P}}^2}{8\pi} \frac{V'}{V} \tag{2.63}$$

$$= -6 \, \frac{M_{\mathrm{P}}^2}{16\pi} \left(\frac{V'}{V} \right)^2 + 2 \, \frac{M_{\mathrm{P}}^2}{8\pi} \left(\frac{V''}{V} \right) \tag{2.64}$$

$$= -6\epsilon + 2\eta \,. \tag{2.65}$$

Finally, an expression may be found for the tensor spectral index, n_{T}. From its definition in Eq. (2.45),

$$
\begin{aligned}
n_{\mathrm{T}} &= \frac{\mathrm{d}\ln A_{\mathrm{T}}^2}{\mathrm{d}\ln k} && (2.66)\\
&= \frac{\mathrm{d}\ln A_{\mathrm{T}}^2}{\mathrm{d}\phi}\frac{\mathrm{d}\phi}{\mathrm{d}\ln k} && (2.67)\\
&\simeq \frac{\mathrm{d}\ln V}{\mathrm{d}\phi}\frac{\mathrm{d}\phi}{\mathrm{d}\ln k} && (2.68)\\
&= -\frac{V'}{V}\frac{M_{\mathrm{P}}^2}{8\pi}\frac{V'}{V} && (2.69)\\
&= -2\epsilon\,. && (2.70)
\end{aligned}
$$

Hence we have a 'consistency' equation,

$$R = -8n_{\mathrm{T}}\,. \tag{2.71}$$

2.2.4 Cosmic microwave background

In 1965, Penzias and Wilson made a discovery of great significance to cosmology: the *cosmic microwave background radiation* (CMB). This takes the form of microwave radiation which we observe coming from all directions in the sky, exhibiting a nearly perfect black-body spectrum with characteristic temperature of 2.73 K. Such radiation is characteristic of a system which is in thermal equilibrium. Black-body radiation also has the property that it remains as black-body radiation as the Universe expands—although the temperature decreases. (The energy density of black-body radiation of temperature T obeys $\epsilon \propto T^4$, and, in a radiation dominated Universe, $\epsilon \propto 1/a^4$. Hence, $T \propto 1/a$.) So the CMB points to a period when the Universe was in thermal equilibrium at a higher temperature. It can be worked out that the CMB would have been formed about 350 000 years after the Big Bang, when the Universe was in thermal equilibrium at approximately 3000 K (Liddle, 2003, pp. 79, 87). (The Universe would have been roughly one thousandth of its current size, since $a \propto 1/T$, and hence the scale factor would have been $2.73/3000 \simeq 1/1000$ of its value today.) This corresponds to the period of **recombination**, when the energy of the radiation dropped sufficiently to allow electrons to combine with protons to form atoms. The Universe changed from being opaque to being transparent as the radiation 'decoupled' from the matter, and the photons began to travel freely through space. It is these photons which we now observe in the CMB. The CMB is also called the **surface of last scattering**, since we observe it as a surface, and since it was formed at the time when the photons were last scattered by the matter in the Universe.

Although the CMB is highly isotropic, there are tiny anisotropies, first discovered in 1992 by the NASA's COBE (COsmic Background Explorer) satellite. The anisotropies are equivalent to a variation in the characteristic temperature of only

$$\frac{\Delta T}{T} \sim 10^{-5}\,. \tag{2.72}$$

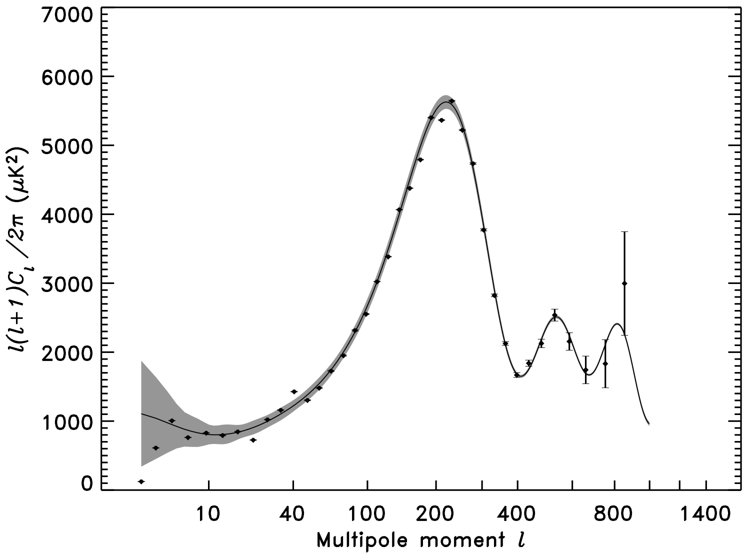

Figure 2.3: Angular power spectrum of the CMB, from the WMAP Science Team (Hinshaw et al., 2003). The data are plotted with one-sigma errors. The solid line shows the best-fit model, with the grey band showing the one-sigma uncertainty in this. The multipole moment, ℓ, corresponds to an angular scale of approximately $180°/\ell$. Hence the first peak corresponds to an angular scale of roughly one degree.

It is these 'ripples' which are thought to reflect the inhomogeneities in the early Universe which eventually led to the formation of stars and galaxies. They also allow us to test predictions of inflationary models.

The signatures of the early Universe are seen in the CMB using the angular power spectrum, which shows how the size of the anisotropies varies with different angular scales. Fig. 2.3 shows the angular power spectrum from NASA's recent Wilkinson Microwave Anisotropy Probe (WMAP) mission[2].

The angular power spectrum is found by decomposing the anisotropies into **spherical harmonics**, $Y_{\ell m}(\theta, \phi)$, the analogue of a Fourier series for the surface of a sphere. This is used to define the multipoles, $a_{\ell m}$, of the anisotropy, (Liddle and Lyth, 2000, p. 116)

$$\frac{\Delta T}{T}(\theta, \phi) = \sum_{\ell m} a_{\ell m} Y_{\ell m}(\theta, \phi) \,. \tag{2.73}$$

The features we are looking for are independent of orientation, so we take an average of the coefficients over ϕ to obtain the **radiation angular power spectrum**,

$$C_\ell \equiv \left\langle |a_{\ell m}|^2 \right\rangle \,. \tag{2.74}$$

C_ℓ corresponds to an angular scale of approximately $180°/\ell$, and a large value of C_ℓ indicates large anisotropies on that scale. $\ell = 1$ corresponds to the **dipole moment**, due to our motion with respect

[2]`http://map.gsfc.nasa.gov/`

to the CMB. This is usually ignored, as it does not represent intrinsic anisotropy in the CMB, so plots of the power spectrum start at $\ell = 2$, as in Fig. 2.3.

The horizon scale at decoupling would have been $90\,\mathrm{h}^{-1}$ Mpc, corresponding to $\ell \simeq 70$ (Liddle and Lyth, 2000, p. 120). At higher values of the multipole moment, ℓ, i.e., smaller angular scales, the points being compared would have been in causal contact at the time of decoupling. Hence such anisotropies may reflect physical processes taking place at this time. The maximum size of these features would have been the Hubble length at the time of decoupling, which subtends a certain angle to us as we see it in the surface of last scattering. This angle depends most strongly on the geometry of the Universe—whether it is flat, open or closed. This angle is related to the location of the first **acoustic peak** in the power spectrum, which is at $\ell \simeq 200$ in Fig. 2.3, corresponding to an angle of roughly one degree. Recent observations of the CMB using this first peak have suggested that the Universe is very flat: the WMAP team found that $\Omega_{\rm tot} = 1.02 \pm 0.02$ (Bennett et al., 2003).

On the other hand, smaller values of ℓ correspond to structure present in the Universe before decoupling, since the points in the CMB separated by angular scales this large were out of causal contact at the time of decoupling. Probing such large scales enables us to investigate the primordial perturbations in the Universe. COBE made observations on such large scales to high precision. This has enabled a constraint to be placed on the amplitude of scalar perturbations, as has been done by Bunn et al. (1996). The **COBE normalization** is found to be (Liddle and Lyth, 2000, p. 248)

$$A_{\rm S} = 1.94 \times 10^{-5}\,, \tag{2.75}$$

for a scale-invariant spectrum (which is why k does not appear in the expression). They found a more complicated expression in the case of a spectrum which is not scale-invariant, but the normalization is generally very similar to that given, particularly for a flat Universe and a nearly scale-invariant spectrum.

From the WMAP data, combined with other data sets including the 2dF galaxy redshift survey, Leach and Liddle (2003b) have found constraints on inflationary parameters. They relate the spectral tilts, $n - 1$ and $n_{\rm T}$, to their slow-roll parameters, using equations equivalent to Eqs. (2.65) and (2.70). Hence, observations constraining these indexes may be used to test inflationary models. It is this analysis which is used in Chapter 5 to test braneworld models of inflation, where the observational constraints used in the plots are taken from that paper.

Chapter 3

Braneworld Inflation

3.1 Cosmology on the brane

3.1.1 The braneworld

Physics in the twentieth century had a problem. Two of the most important theories in physics—quantum mechanics and general relativity—were formulated and found to be contradictory. General relativity describes a universe which is very smooth (if not flat), and quantum mechanics suggests that things are very 'lumpy' on small scales. These problems have been (sketchily) resolved by new theories: **superstring theory** (or string theory) and the related and more recent **M-theory**, which seeks to unify various string theories. Essentially the difficulties with quantum mechanics and general relativity vanish when one takes into consideration the spatial extent of the strings, which are believed to be the fundamental constituent of the Universe. It turns out to be impossible to make measurements on the scales on which things would appear 'lumpy' (that is, below the Planck length, 1.6×10^{-35}m). In the words of Brian Greene,

> But the whole conflict between general relativity and quantum mechanics arises from the sub-Planck-length properties of the spatial fabric. *If the elementary constituent of the universe cannot probe sub-Planck-scale distances, then neither it nor anything made from it can be affected by the supposedly disastrous short-distance quantum undulations.* ... Therefore ... one can even say that the supposed tempestuous sub-Planckian quantum undulations *do not exist.* (Greene, 1999, p. 156, emphasis in original)

However, these theories introduce new complications to the Universe: **extra dimensions**. The reason is that they are *supersymmetric*, meaning that they are symmetric not only regarding position, velocity and gravity, but also regarding the quantum mechanical *spin* of particles. Supersymmetric theories (e.g., superstring theory—hence the name) are naturally formulated in more than four dimensions. String theory gives the Universe ten dimensions in total (nine spatial and one time) and

M-theory has an eleven-dimensional Universe (ten spatial, and one time).

The idea that there are more dimensions than the familiar four (one time and three spatial) was not new when string theory was being developed towards the end of the twentieth century. It had in fact been proposed in 1919 by Kaluza, who, in a letter to Einstein, found that formulating general relativity in five dimensions (rather than four) led to an elegant way of combining of general relativity with Maxwell's electromagnetic theory. This research was published by Kaluza (1921), and was developed by Klein (1926), but, after meeting some serious problems, research into extra dimensions largely ceased. The extra dimensions in the work of Kaluza and Klein, as well as in string theory and M-theory, are considered to be very small, or *compact*. This is analogous to a hose-pipe, which has an extended dimension—its length—and a compact dimension—its circumference.

However, it has recently been suggested (Arkani-Hamed et al., 1998; Antoniadis et al., 1998) that if the ordinary matter fields are confined to the three usual extended dimensions only, with gravity allowed to pass through all of the dimensions, then the extra dimensions need not be so small, and may be very large, even infinite. The three-dimensional space on which the matter exists is termed the **brane**, and the higher-dimensional space is called the **bulk**.

During the last few years, the case with one of the extra dimensions being large (perhaps unbounded) has received a significant amount of attention. This leads to a Universe of five significant dimensions (four spatial, and one time). Randall and Sundrum (1999a) proposed a model with two branes, one of which contains the standard model fields. They then proposed a second model (Randall and Sundrum, 1999b), which has a single brane. (This is equivalent to the first model, but with one of the original branes removed to infinity.) This is the **Randall–Sundrum type II braneworld**, hereafter called the *braneworld scenario*, which forms the basis for the rest of this thesis. It is illustrated pictorially in Fig. 3.1.

3.1.2 Braneworld cosmology

As would be expected, the braneworld scenario modifies the Einstein equations for gravity on the brane. This is because the gravitational fields are allowed to propagate through an additional large dimension. The equations of general relativity must be investigated in this five-dimensional case, to enable us to predict the form which gravity will take on the brane. A crucial question will be whether the resulting equations allow us to recover the familiar four-dimensional (effective) Einstein gravity on the brane, for all ordinary situations. If not, we must reject this braneworld scenario, since the standard four-dimensional gravitational equations have been tested to high precision.

The cosmological equations on the brane have been derived using two methods (see Brax and van de Bruck, 2003). Shiromizu et al. (2000) derive the effective four-dimensional gravitational equations without any specific assumptions about the brane, and then investigate these equations in particular braneworld models. They recover the standard Einstein equations in the low-energy limit. In the special case where the bulk is exactly anti-de Sitter, i.e., has a negative cosmological constant

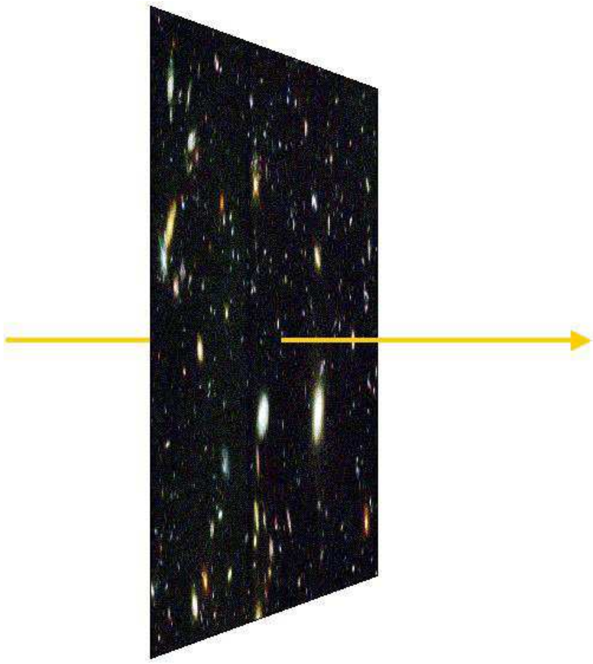

Figure 3.1: The Randall–Sundrum type II braneworld. Our Universe is confined to a 'brane' with three spatial dimensions (shown as two here) with an additional large dimension, shown by the arrow. The full spacetime, with four large dimensions and one time dimension, is called the 'bulk'.

only, they find that the matter on the brane will be spatially homogeneous. The other method is simpler, having more restrictions from the outset, but leads to results in agreement with the first. It also leads to a 'Friedmann equation' which reduces to the standard Friedmann equation at low energies. I will present a brief description of this second method here, following Binétruy et al. (2000b).

The fifth dimension is labelled by y, and the brane is located at $y = 0$. The five-dimensional metric is assumed to be of the form,

$$\mathrm{d}s^2 = \tilde{g}_{AB}\mathrm{d}x^A\mathrm{d}x^B \tag{3.1}$$

$$= g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + b^2\mathrm{d}y^2 \tag{3.2}$$

$$= -n^2(\tau,y)\mathrm{d}\tau^2 + a^2(\tau,y)\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j + b^2(\tau,y)\mathrm{d}y^2\,, \tag{3.3}$$

where upper case italic indices run from 0 to 4, Greek indices run from 0 to 4, and lower case italic indices run from 1 to 3. γ_{ij} is maximally symmetric (as in the Robertson–Walker metric). This is the metric for a homogeneous and isotropic Universe on the brane.

In order to obtain the equations for the evolution of a, the scale factor, the Einstein equations need to be solved for this metric and some energy-momentum tensor. The Einstein equations in five dimensions take the usual form,

$$\tilde{G}_{AB} = \kappa^2\tilde{T}_{AB}\,. \tag{3.4}$$

We take the bulk to be anti-de Sitter spacetime, having a negative cosmological constant, Λ_5. Hence the bulk has pressure, $p_{\mathrm{B}} = -\rho_{\mathrm{B}} = -\Lambda_5$. The energy-momentum tensor is therefore taken to be

$$T^A{}_B = \mathrm{diag}\left[(-\rho_{\mathrm{b}}, p_{\mathrm{b}}, p_{\mathrm{b}}, p_{\mathrm{b}}, 0)\frac{\delta(y)}{b} + (-\Lambda_5, -\Lambda_5, -\Lambda_5, -\Lambda_5, -\Lambda_5)\right] , \tag{3.5}$$

where ρ_{b} and p_{b} are the energy density and pressure on the brane respectively.

Einstein's equations with this energy-momentum tensor, along with the metric shown above, lead to the following equation,

$$\left(\frac{\dot{a}}{na}\right)^2 = \frac{1}{6}\kappa^2\Lambda_5 + \left(\frac{a'}{ba}\right)^2 - \frac{k}{a^2} + \frac{\mathcal{C}}{a^4} , \tag{3.6}$$

where a prime denotes differentiation with respect to y and $\mathcal{C}$ is a constant of integration. On the brane, one needs to worry about the discontinuity. This is dealt with using certain junction conditions (Binétruy et al., 2000a), leading to an equation for the brane,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\kappa^2}{6}\Lambda_5 + \frac{\kappa^4}{36}\rho_{\mathrm{b}}^2 + \frac{\mathcal{C}}{a^4} - \frac{k}{a^2} , \tag{3.7}$$

where n has been set to be equal to 1, by a suitable transformation to the time variable.

If the brane is allowed to possess a constant energy density, λ, called the **brane tension**, as well as ordinary matter, we have,

$$\rho_{\mathrm{b}} = \rho + \lambda . \tag{3.8}$$

This gives

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\kappa^2}{6}\Lambda_5 + \frac{\kappa^4}{36}\lambda^2 + \frac{\kappa^4}{18}\lambda\rho + \frac{\kappa^4}{36}\rho^2 + \frac{\mathcal{C}}{a^4} - \frac{k}{a^2} . \tag{3.9}$$

The first two terms on the right hand side behave like an effective cosmological constant on the brane, which we may define as (Shiromizu et al., 2000),

$$\frac{\Lambda_4}{3} \equiv \frac{\kappa^2}{6}\Lambda_5 + \frac{\kappa^4}{36}\lambda^2 . \tag{3.10}$$

If we also write

$$\frac{8\pi}{M_4^2} \equiv 8\pi G \equiv \frac{\kappa^4\lambda}{6} , \tag{3.11}$$

where M_4 is the effective Planck scale and G is Newton's constant on the brane, we obtain a 'Friedmann equation' of the form,

$$H^2 = \frac{\Lambda_4}{3} + \frac{8\pi}{3M_4^2}\,\rho\left(1 + \frac{\rho}{2\lambda}\right) + \frac{\mathcal{C}}{a^4} - \frac{k}{a^2} . \tag{3.12}$$

Binétruy et al. (2000b) also find that energy is conserved on the brane, i.e., that the fluid equation is still valid:

$$\rho_{\mathrm{b}} + 3\frac{\dot{a}}{a}(\rho_{\mathrm{b}} + p_{\mathrm{b}}) = 0 . \tag{3.13}$$

The fundamental Planck scale, M_5, is related to the coupling constant, κ, by (Binétruy et al., 2000b)

$$\kappa^2 = \frac{8\pi^2}{M_5^3} , \tag{3.14}$$

so (Maartens et al., 2000)

$$M_4 = \sqrt{\frac{3}{4\pi}} \left(\frac{M_5^2}{\sqrt{\lambda}} \right) M_5 \,. \tag{3.15}$$

Thus the fundamental Planck scale is different to the four-dimensional effective Planck scale, $M_4 = 1.2 \times 10^{19}$ GeV. In order to get an order-of-magnitude estimate for its value, we need a constraint on λ. We must have the standard cosmological expansion for temperatures less than roughly 1 MeV, or nucleosynthesis will be affected. As will be seen below, this implies that we have $\rho \ll \lambda$ for such energies. The energy density is related to the temperature by $\rho \simeq T^4$ (cf. Liddle and Lyth, 2000, p. 19), so $\lambda \gg \rho \simeq (1\,\text{MeV})^4 = 10^{-12}\,(\text{GeV})^4$ (Copeland et al., 2001). Hence

$$\begin{aligned} M_5^3 &= \sqrt{\frac{4\pi}{3}} \sqrt{\lambda} M_4 & (3.16)\\ &\gg \sqrt{\frac{4\pi}{3}} 10^{-6}\,(\text{GeV})^2\,10^{19}\,\text{GeV}\,, & (3.17)\\ M_5 &\gg 10^4\,\text{GeV}\,. & (3.18) \end{aligned}$$

The fundamental Planck scale may therefore be significantly below the effective Planck scale. This could provide a solution to the **hierarchy problem** (Arkani-Hamed et al., 1998): the puzzle as to why the Planck scale and the Weak scale differ by some 16 orders of magnitude. A smaller fundamental Planck scale would alleviate this discrepancy.

The four-dimensional cosmological constant is usually fine-tuned to be zero in the braneworld scenario. This is necessary in order to produce static solutions (Brax and van de Bruck, 2003), and is achieved by balancing Λ_5 and λ to satisfy

$$0 = \Lambda_5 + \frac{\kappa^2}{6} \lambda^2 \,. \tag{3.19}$$

Small deviations from this fine-tuning can lead to a small effective cosmological constant which would dominate in later periods of the evolution of the Universe, as favoured by recent observations. This fine-tuning is the braneworld version of the cosmological constant problem, i.e., the problem of why the cosmological constant takes the particular value it is observed to have.

The $\mathcal{C}/a^4$ term is known as the **dark radiation**, since it decays in the same way as radiation (the energy density during radiation domination is proportional to a^{-4}). If the bulk spacetime is anti-de Sitter, then this term is equal to zero (Brax and van de Bruck, 2003). However, as we shall see, this term may be neglected during inflation.

Neglecting any dark radiation, Eq. (3.12) reduces to the usual Friedmann equation when $1 \gg \rho/2\lambda$, i.e. at low energies. So in order to discover whether this is a realistic description of the Universe, high-energy situations must be investigated.

The energy density, ρ would have been much higher in the early Universe. So by investigating signatures of physical processes at work in the early Universe, it may be possible to detect braneworld effects. It therefore makes sense to consider *inflation on the brane*. During such a rapid period of expansion, the final two terms in equation (3.12) would quickly become negligible, as the scale factor, a, increases rapidly. Λ_4 is also taken to be negligible. This gives rise to a 'Friedmann equation' of the form,

$$H^2 = \frac{8\pi}{3M_4^2}\,\rho\left(1+\frac{\rho}{2\lambda}\right), \tag{3.20}$$

which will form the basis of our investigations of braneworld inflation.

3.2 Inflation on the brane

At low energies, i.e., when $\rho \ll \lambda$, inflation in the braneworld scenario behaves in exactly the same way as standard inflation. But at higher energies we would expect the dynamics of inflation to be changed. In this section, the details of this modification will be investigated. In order to do this, we will follow the derivations in Section 2.2 but with the modified Friedmann equation (3.20).

3.2.1 Dynamics of inflation

We first find the condition required for inflation to take place. The fluid equation, (3.13), still holds, so we have,

$$\dot{\rho} = -3H(\rho+p)\,, \tag{3.21}$$

where ρ and p are the energy density and pressure respectively of the scalar field, defined in Eqs. (2.10) and (2.11). Differentiating Eq. (3.20) with respect to time gives,

$$2H\dot{H} = 2H\left(\frac{\ddot{a}}{a}-\frac{\dot{a}^2}{a^2}\right) = \frac{8\pi}{3M_4^2}\left[\dot{\rho}+\dot{\rho}\frac{\rho}{\lambda}\right] \tag{3.22}$$

$$= -\frac{8\pi H}{M_4^2}(\rho+p)\left(1+\frac{\rho}{\lambda}\right), \tag{3.23}$$

by the fluid equation. Hence we obtain an acceleration equation,

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{M_4^2}(\rho+p)\left(1+\frac{\rho}{\lambda}\right)+H^2 \tag{3.24}$$

$$= -\frac{4\pi}{M_4^2}\left[(\rho+p)\left(1+\frac{\rho}{\lambda}\right)-\frac{2}{3}\rho\left(1+\frac{\rho}{2\lambda}\right)\right] \tag{3.25}$$

$$= -\frac{4\pi}{3M_4^2}\left[(\rho+3p)+\frac{\rho}{\lambda}(2\rho+3p)\right]. \tag{3.26}$$

During inflation, $\ddot{a} > 0$, so the condition for inflation is

$$0 < -[\lambda(\rho+3p)+\rho(2\rho+3p)]\,, \tag{3.27}$$

$$p < -\frac{\lambda\rho+2\rho^2}{3(\lambda+\rho)} \tag{3.28}$$

$$= -\frac{\rho}{3}\left(\frac{\lambda+2\rho}{\lambda+\rho}\right), \tag{3.29}$$

as found by Maartens et al. (2000). For $\rho \ll \lambda$, this reduces to the standard condition for inflation, $p < -\rho/3$, as in Eq. (2.17). For $\rho \gg \lambda$, we have the stronger condition,

$$p < -\frac{2\rho}{3}\,. \tag{3.30}$$

As there is no qualitative change in the definition of inflation, we define the **slow-roll conditions** as in Eqs. (2.18) and (2.19):

$$\frac{1}{2}\dot{\phi}^2 \ll V\,, \tag{3.31}$$

$$|\ddot{\phi}| \ll |3H\dot{\phi}|\,, \tag{3.32}$$

so that inflation is considered to be taking place when these conditions are satisfied. As before, we therefore find two equations which describe the dynamics of slow-roll inflation (where approximate equality denotes equality under the slow-roll conditions),

$$H^2 \simeq \frac{8\pi}{3M_4^2} V \left(1 + \frac{V}{2\lambda}\right)\,, \tag{3.33}$$

$$3H\dot{\phi} \simeq -V'\,. \tag{3.34}$$

These two equations (combined with their time derivatives, assuming that $\dddot{\phi}$ is small) enable the slow-roll conditions (3.31) and (3.32) to be written respectively as

$$\frac{M_4^2}{16\pi}\left(\frac{V'}{V}\right)^2 \frac{1}{1+V/2\lambda} \ll 1\,, \tag{3.35}$$

$$\left|\frac{M_4^2}{8\pi}\frac{V''}{V}\frac{1}{1+V/2\lambda}\right| \ll 1\,. \tag{3.36}$$

Following Maartens et al. (2000), we define the slow-roll parameters on the brane as follows:

$$\epsilon \equiv \frac{M_4^2}{16\pi}\left(\frac{V'}{V}\right)^2 \frac{1+V/\lambda}{(1+V/2\lambda)^2}\,, \tag{3.37}$$

$$\eta \equiv \frac{M_4^2}{8\pi}\frac{V''}{V}\frac{1}{1+V/2\lambda}\,, \tag{3.38}$$

which reduce to the standard slow-roll parameters in the low-energy limit. These parameters are taken from the slow-roll conditions, except that ϵ is slightly different from the expression in Eq. (3.35)—a change which amounts to increasing it by a factor no greater than 2. The reason for this alteration is to retain the same equation for n in terms of the slow-roll parameters as in the standard cosmology, as shown in Eq. (3.65). Slow-roll inflation implies that $\epsilon \ll 1$ and $|\eta| \ll 1$, as in the standard cosmology.

The number of e-foldings gains an extra factor (shown in parentheses) from Eq. (3.33) to give (Maartens et al., 2000)

$$N \simeq -\frac{8\pi}{M_4^2}\int_{\phi_{\rm i}}^{\phi_{\rm f}} \frac{V}{V'}\left(1+\frac{V}{2\lambda}\right) {\rm d}\phi\,, \tag{3.39}$$

where $\phi_{\rm i}$ and $\phi_{\rm f}$ are the values of the scalar field at the beginning and end of the expansion respectively.

3.2.2 Growth of structure

There is no change in the equation for the spectrum of scalar perturbations in terms of H, which remains as Eq. (2.46),

$$A_{\rm S}^2 = \frac{4}{25}\left[\left(\frac{H}{\dot\phi}\right)\left(\frac{H}{2\pi}\right)\right]^2_{k=aH} \tag{3.40}$$

$$\simeq \frac{512\pi}{75M_4^6}\frac{V^3}{V'^2}\left(1+\frac{V}{2\lambda}\right)^3, \tag{3.41}$$

under the slow-roll approximation (Maartens et al., 2000). For the tensor perturbations, however, there is a change from Eq. (2.50). The equation for their spectrum becomes (Langlois et al., 2000; Huey and Lidsey, 2001)

$$A_{\rm T}^2 = \frac{4}{25\pi M_4^2}H^2F^2(H/\mu)\bigg|_{k=aH} \tag{3.42}$$

$$\simeq \frac{32}{75M_4^4}V\left(1+\frac{V}{2\lambda}\right)F^2(H/\mu), \tag{3.43}$$

where

$$F(x) = \left[\sqrt{1+x^2}-x^2\ln\left(\frac{1}{x}+\sqrt{1+\frac{1}{x^2}}\right)\right]^{-1/2} \tag{3.44}$$

$$= \left(\sqrt{1+x^2}-x^2\sinh^{-1}\frac{1}{x}\right)^{-1/2}, \tag{3.45}$$

and μ, the mass scale, is given by

$$\mu = \frac{1}{M_4}\sqrt{\frac{4\pi\lambda}{3}}. \tag{3.46}$$

Comparing the equation for μ with Eq. (3.33) for H^2 shows that

$$x^2 = \frac{H^2}{\mu^2} \simeq \frac{2V}{\lambda}\left(1+\frac{V}{2\lambda}\right). \tag{3.47}$$

The equation for F simplifies in the high- and low-energy limits. In the low-energy limit, $x^2 \simeq 2V/\lambda \ll 1$ and in the high-energy limit, $x^2 \simeq (V/\lambda)^2 \gg 1$. So, in the low-energy limit,

$$F^2 \simeq \left(1-x^2\ln\frac{2}{x}\right)^{-1} \tag{3.48}$$

$$\simeq 1, \tag{3.49}$$

since $y > \ln y$ for positive y, which implies that $2/x > \ln(2/x)$ or $(x/2)\ln(2/x) < 1$ and hence $x^2\ln(2/x) = 2x(x/2)\ln(2/x) < 2x \ll 1$. This is as expected, since in the low-energy limit, the expressions must be the same as those derived without considering the brane-effects. In the high-energy limit, using $\sinh^{-1} y \simeq y - y^3/6$ for small y,

$$F^2 \simeq \left[\sqrt{1+\left(\frac{V}{\lambda}\right)^2} - \left(\frac{V}{\lambda}\right)^2 \sinh^{-1}\frac{\lambda}{V}\right]^{-1} \quad (3.50)$$

$$\simeq \left[\frac{V}{\lambda}\sqrt{1+\left(\frac{\lambda}{V}\right)^2} - \left(\frac{V}{\lambda}\right)^2 \left(\frac{\lambda}{V} - \frac{\lambda^3}{6V^3}\right)\right]^{-1} \quad (3.51)$$

$$\simeq \left[\frac{V}{\lambda}\left(1+\frac{\lambda^2}{2V^2}\right) - \frac{V}{\lambda} + \frac{\lambda}{6V}\right]^{-1} \quad (3.52)$$

$$= \left(\frac{\lambda}{2V} + \frac{\lambda}{6V}\right)^{-1} \quad (3.53)$$

$$= \frac{3V}{2\lambda} \gg 1\,, \quad (3.54)$$

as found by Langlois et al. (2000). This may be expected to increase the amplitude of gravitational waves produced in the high-energy regime, but that is dependent on the value of the scalar field at which this is evaluated. (This is equivalent to an increase in R, since $A_{\rm S}$ is fixed by the COBE normalization.) As we shall see in Chapter 4, for some potentials the amplitude is increased in the high-energy limit, whereas for other potentials, there is actually a decrease in the tensor perturbations.

The ratio of tensor to scalar perturbations is defined as before to be

$$R \equiv 16\,\frac{A_{\rm T}^2}{A_{\rm S}^2}\,, \quad (3.55)$$

which Huey and Lidsey (2001) find to obey the same consistency equation as found previously,

$$R = -8n_{\rm T}\,, \quad (3.56)$$

where the tensor spectral index is defined as in the standard case.

For the scalar spectral index, we may derive an expression under the slow-roll approximation,

$$n-1 = \frac{{\rm d}\ln A_{\rm S}^2}{{\rm d}\ln k} \quad (3.57)$$

$$= \frac{{\rm d}\ln A_{\rm S}^2}{{\rm d}\phi}\frac{{\rm d}\phi}{{\rm d}\ln k} \quad (3.58)$$

$$\simeq \left(\frac{{\rm d}\ln V^3}{{\rm d}\phi} - \frac{{\rm d}\ln V'^2}{{\rm d}\phi} + \frac{{\rm d}\ln(1+V/2\lambda)^3}{{\rm d}\phi}\right)\frac{{\rm d}\phi}{{\rm d}\ln k}\,. \quad (3.59)$$

In the braneworld scenario, we have

$${\rm d}\ln k \simeq -{\rm d}N \quad (3.60)$$

$$\simeq -\frac{8\pi}{M_4^2}\frac{V}{V'}\left(1+\frac{V}{2\lambda}\right){\rm d}\phi\,. \quad (3.61)$$

Hence,

$$n - 1 \simeq -\left(3\,\frac{\mathrm{d}\ln V}{\mathrm{d}\phi} - 2\,\frac{\mathrm{d}\ln V'}{\mathrm{d}\phi} + 3\frac{\mathrm{d}\ln(1+V/2\lambda)}{\mathrm{d}\phi}\right)\frac{M_4^2}{8\pi}\frac{V'}{V}\frac{1}{1+V/2\lambda} \tag{3.62}$$

$$= -\left(3\,\frac{V'}{V} - 2\,\frac{V''}{V'} + 3\frac{V'/2\lambda}{1+V/2\lambda}\right)\frac{M_4^2}{8\pi}\frac{V'}{V}\frac{1}{1+V/2\lambda} \tag{3.63}$$

$$= -6\,\frac{M_4^2}{16\pi}\left(\frac{V'}{V}\right)^2\left(\frac{1}{1+V/2\lambda} + \frac{V/2\lambda}{(1+V/2\lambda)^2}\right) + 2\,\frac{M_4^2}{8\pi}\left(\frac{V''}{V}\right)\frac{1}{1+V/2\lambda} \tag{3.64}$$

$$= -6\epsilon + 2\eta\,, \tag{3.65}$$

as before.

The braneworld scenario, we have found, is a plausible description of the Universe. At low-energies, such as we experience today, the resulting cosmology is identical to the standard cosmology. During the early Universe, particularly during inflation, there may be changes to the perturbations predicted by the standard cosmology, if the energy density is sufficiently high compared with the brane tension. The task for the rest of this thesis will be to obtain predictions for the perturbations arising from some specific models of inflation, and to try to ascertain whether these predictions are compatible with observational constraints.

Chapter 4

Perturbations from Monomial Potentials

Having put in place the required equations, we now turn to consider perturbations arising from particular potentials. The potential may essentially take any form, but we will restrict our attention to **monomial** potentials, i.e., those of the form

$$V = m\phi^{\alpha}\,, \tag{4.1}$$

where α and m are constants. The exponent, α, is usually an even integer, and we will restrict our attention to $\alpha \geq 2$. The aim will be to predict $n-1$, the scalar spectral index, and R, the relative amplitude of tensor to scalar perturbations.

With the above potential, the slow-roll parameters may be found from Eqs. (3.37) and (3.38) to be

$$\epsilon = \frac{M_4^2}{16\pi}\left(\frac{\alpha}{\phi}\right)^2 \frac{1+m\phi^{\alpha}/\lambda}{(1+m\phi^{\alpha}/2\lambda)^2}\,, \tag{4.2}$$

$$\eta = \frac{M_4^2}{8\pi}\,\frac{\alpha(\alpha-1)}{\phi^2}\,\frac{1}{1+m\phi^{\alpha}/2\lambda}\,. \tag{4.3}$$

The brane tension, λ, may take any value, so the slow-roll parameters satisfy

$$\frac{1}{2}\eta \leq \epsilon \leq 2\eta\,, \tag{4.4}$$

with $\eta/2 \leq \epsilon$ in the low-energy limit.

Inflation ends when the slow-roll conditions, $\epsilon \ll 1$ and $|\eta| \ll 1$, no longer both hold. Usually, inflation is taken to end when $\epsilon = 1$ (see Section 2.2.2), but, for ease of computation, we will take $\eta = 1$ to mark the end of inflation. This does affect the results, but the differences are negligible in all that follows.

The number of e-foldings, from Eq. (3.39), becomes

$$N \simeq -\frac{8\pi}{M_4^2}\int_{\phi_N}^{\phi_{\rm end}} \frac{\phi}{\alpha}\left(1+\frac{m\phi^{\alpha}}{2\lambda}\right)\mathrm{d}\phi\,, \tag{4.5}$$

where ϕ_N is the value of the inflaton field N e-foldings before the end of inflation, and $\phi_{\rm end}$ is its value at the end of inflation.

4.1 Low- and high-energy limits

The equations simplify significantly in the low- and high-energy limits, in which we may obtain expressions for $n-1$ and R (under the slow-roll approximation) which are independent of λ, m and M_4.

In the **low-energy limit**, the final term from Eq. (4.3) vanishes, so $\eta = 1$ implies that

$$\phi_{\rm end}^2 = \frac{M_4^2 \alpha(\alpha - 1)}{8\pi} \,. \tag{4.6}$$

The term in parentheses in Eq. (4.5) also vanishes, giving

$$N = \frac{4\pi}{M_4^2 \alpha} (\phi_N^2 - \phi_{\rm end}^2) \,. \tag{4.7}$$

Hence,

$$\phi_N^2 = \left(N + \frac{\alpha - 1}{2} \right) \frac{M_4^2 \alpha}{4\pi} \,. \tag{4.8}$$

This value of ϕ is used to evaluate the slow-roll parameters, which are then substituted into $n - 1 = -6\epsilon + 2\eta$ and $R = 16\epsilon$ (the low-energy approximation) to give the desired results, shown below.

In the **high-energy limit**, we have $m\phi^\alpha / 2\lambda \gg 1$. In this case, $\eta = 1$ implies that

$$\phi_{\rm end}^{\alpha+2} = \frac{M_4^2 \alpha(\alpha - 1)}{4\pi} \frac{\lambda}{m} \,. \tag{4.9}$$

The equation for the number of e-foldings becomes

$$N = \frac{4\pi}{M_4^2 \alpha(\alpha + 2)} \frac{m}{\lambda} (\phi_N^{\alpha+2} - \phi_{\rm end}^{\alpha+2}) \,. \tag{4.10}$$

Hence,

$$\phi_N^{\alpha+2} = \left(N + \frac{\alpha - 1}{\alpha + 2} \right) \frac{M_4^2 \alpha(\alpha + 2)}{4\pi} \frac{\lambda}{m} \,. \tag{4.11}$$

In the high-energy limit, ϵ becomes

$$\epsilon = \frac{M_4^2}{4\pi} \frac{\lambda V'^2}{V^3} = \frac{M_4^2}{4\pi} \frac{\alpha^2}{\phi^{\alpha+2}} \frac{\lambda}{m} \,, \tag{4.12}$$

and R, from Eqs. (3.43), (3.54) and (3.41), is given by

$$R = 16 \frac{A_{\rm T}^2}{A_{\rm S}^2} \tag{4.13}$$

$$= 16 \frac{\frac{32}{75 M_4^4} \frac{V^2}{2\lambda} \frac{3V}{2\lambda}}{\frac{512\pi}{75 M_4^6} \frac{V^3}{V'^2} \frac{V^3}{(2\lambda)^3}} \tag{4.14}$$

$$= \frac{6 M_4^2}{\pi} \frac{\lambda V'^2}{V^3} \tag{4.15}$$

$$= 24\epsilon \,. \tag{4.16}$$

As in the low-energy case, the expression for $\phi_N^{\alpha+2}$ may be used to obtain the slow-roll parameters, and hence $R = 24\epsilon$ and $n - 1 = -6\epsilon + 2\eta$.

Table 4.1: low- and high-energy limits for scalar spectral index, n, and ratio of tensor to scalar perturbations, R, for potentials $V \propto \phi^{\alpha}$. The end of inflation is defined by $\eta = 1$ and the number of e-foldings is taken to be 55.

α	$n_{\rm low} - 1$	$n_{\rm high} - 1$	$R_{\rm low}$	$R_{\rm high}$
2	-0.036	-0.045	0.144	0.217
4	-0.053	-0.054	0.283	0.288
6	-0.070	-0.058	0.417	0.324
8	-0.085	-0.061	0.547	0.345

The expressions for $n-1$ and R in **both limits** are therefore:[1]

$$n_{\rm low} - 1 = -\frac{\alpha+2}{2N-1+\alpha}, \tag{4.17}$$

$$n_{\rm high} - 1 = -\frac{4\alpha+2}{(2+\alpha)N-1+\alpha}, \tag{4.18}$$

$$R_{\rm low} = \frac{8\alpha}{2N-1+\alpha}, \tag{4.19}$$

$$R_{\rm high} = \frac{24\alpha}{(2+\alpha)N-1+\alpha}. \tag{4.20}$$

In Table 4.1, numerical results in these limits are shown for a variety of potentials. This is plotted in Fig. 5.2, which shows the high- and low-energy limits for potentials of exponent $\alpha \geq 2$, and enables these predictions to be compared with observational constraints.

In the limit as α tends to infinity, in the high-energy regime the scalar spectral index tends to

$$n_{\rm high} - 1 = \frac{4}{N+1}, \tag{4.21}$$

which corresponds to **steep inflation** driven by an exponential potential (Copeland et al., 2001). In this model, the potential takes the form

$$V(\phi) = V_0 \exp\left(-\sqrt{8\pi}\alpha\frac{\phi}{M_4}\right), \tag{4.22}$$

where α is a constant. This is also called **power law inflation**, since (in the low-energy limit) the scale-factor obeys a power law, $a \propto t^{1/\alpha^2}$. In the standard inflationary scenario, this potential will only produce inflation if $\alpha^2 < 2$. However, Copeland et al. (2001) find that the braneworld scenario can support inflation for any value of α, and also that the resulting perturbations are *independent* of the brane tension, λ (as long as inflation takes place entirely in the high-energy regime). Inflation ends when $\epsilon = 1$ and is followed by reheating, which takes place by gravitational particle production. An alternative method of reheating for steep inflation has recently been proposed, and will be discussed in Chapter 5.

[1] Taking $\epsilon = 1$ to be the end of inflation produces similar results, except that in the low-energy case, the denominator becomes $(2N + \alpha/2)$, and in the high-energy case, the denominator becomes $([2+\alpha]N + \alpha)$. They are hence unchanged for a quadratic potential in the low-energy limit, where $\eta = \epsilon$.

4.2 Number of *e*-foldings

Before being able to find results for any value of the brane tension, and not merely in the high- and low-energy limits, it is necessary to fix N, the number of e-foldings.

Liddle and Leach (2003) considered the number of e-foldings before the end of inflation at which observable perturbations were generated. This is related to $N_{\rm hor}$, the number of e-foldings corresponding to the era at which the comoving Hubble length took the same value that it takes today. At that time, the comoving scale, k, was equal to the present Hubble scale, $a_0 H_0$. They found a plausible upper limit on $N_{\rm hor}$ equal to 60, with that limit raised to 64 in the special case of a quartic potential. The number of e-foldings, N, relating to the perturbations of interest is roughly 4 below $N_{\rm hor}$, as will be discussed below.

It may be asked whether N is changed by much when braneworld effects become significant. From Eq. (7) of Liddle and Leach (2003) we have, assuming instantaneous reheating, in the standard cosmology,

$$N_{\rm hor}^{\rm max} = \frac{1}{4} \ln \frac{\rho_{\rm eq}}{V_{\rm hor}} + \ln \sqrt{\frac{8\pi V_{\rm hor}}{3M_4^2}} \frac{1}{H_{\rm eq}} + \ln 219\, \Omega_0 h \, . \qquad (4.23)$$

In the braneworld cosmology, as a ‘worst-case scenario’ we consider the high-energy regime continuing until matter-radiation equality. Then the actual braneworld effects will change the number of e-foldings by no more than in this case. (In actual fact, the high-energy regime must be over by nucleosynthesis, which took place much earlier. Matter-radiation equality is chosen to simplify the calculations.) Taking the changes into account, Eq. (4.23) still holds, except for an extra term under the square root, from Eq. (3.33) with $V \gg \lambda$. (Although the expansion during the radiation era becomes $a \propto t^{1/4}$ if it takes place in the high-energy regime—rather than the usual $a \propto t^{1/2}$—it is the density as a function of scale factor that enters the equations: this remains unchanged as $\rho \propto a^{-4}$.) In the high-energy case, we therefore have

$$N_{\rm hor}^{\rm max} = \frac{1}{4} \ln \frac{\rho_{\rm eq}}{V_{\rm hor}} + \ln \sqrt{\frac{8\pi V_{\rm hor}}{3M_4^2} \frac{V_{\rm hor}}{2\lambda}} \frac{1}{H_{\rm eq}} + \ln 219\, \Omega_0 h \, , \qquad (4.24)$$

This leads to a maximum number of e-foldings of

$$N_{\rm hor}^{\rm max} = 68.5 + \frac{1}{4} \ln \frac{V_{\rm hor}}{M_4^4} \left(\frac{V_{\rm hor}}{2\lambda} \right)^2 , \qquad (4.25)$$

which differs from Eq. (8) of Liddle and Leach (2003) only in the term in parentheses. Next, from Eq. (3.37), in the high-energy limit, ϵ gains a factor of $(4\lambda/V)$. The perturbation amplitude, $\mathcal{P}_{\rm S,0}$, (which is the same as $\mathcal{P}_{\mathcal{R}} = 25A_{\rm S}^2/4$, cf. Eq. [3.41]) gains a factor of $(V/2\lambda)^3$, so these combine to give

$$\mathcal{P}_{\rm S,0} = \frac{16V}{3M_4^4} \frac{1}{\epsilon} \left(\frac{V}{2\lambda} \right)^2 , \qquad (4.26)$$

$$\epsilon = \frac{M_4^2}{16\pi} \left(\frac{V'}{V} \right)^2 \left(\frac{4\lambda}{V} \right) , \qquad (4.27)$$

which correspond to Eqs. (9) and (10) of Liddle and Leach (2003), differing only in the final terms in parentheses, and an extra factor of 2 in the first equation. With $\mathcal{P}_{\mathrm{S},0} \simeq 2.6 \times 10^{-9}$ (Leach and Liddle, 2003a), i.e., the COBE normalization, this gives,

$$N_{\mathrm{hor}}^{\mathrm{max}} = 68.5 + \frac{1}{4}\ln\frac{3}{16}\,\mathcal{P}_{\mathrm{S},0}\,\epsilon \tag{4.28}$$

$$= 68.5 + \frac{1}{4}\ln\left(\frac{3}{16}\times 2.6\times 10^{-9}\right) + \frac{1}{4}\ln\epsilon \tag{4.29}$$

$$= 63.1 + \frac{1}{4}\ln\epsilon\,, \tag{4.30}$$

which is very similar to Eq. (11) in Liddle and Leach (2003), where the low-energy result is given as

$$N_{\mathrm{hor}}^{\mathrm{max}} = 63.3 + \frac{1}{4}\ln\epsilon\,. \tag{4.31}$$

It is found from results to be derived later on that ϵ takes values between 0.0090 and 0.0101 for the quadratic potential, and between 0.0120 and 0.0184 for the quadratic potential, over the whole range of values of λ. Hence $\ln\epsilon$ is virtually unchanged between the standard cosmology and the high-energy case, so it follows that the plausible upper limit on the number of e-foldings required is essentially the same in both the standard and braneworld cosmologies.

From the same paper, Liddle and Leach find that we also need to take into account a reduction of energy density at the end of inflation, and a reduction in energy density during reheating. These, respectively, would increase the number of e-foldings by a small amount (not likely to be much more than one) and decrease the number of e-foldings by roughly 5 ± 5. With the above values of ϵ, which give $\frac{1}{4}\ln\epsilon \simeq -1$, we have,

$$N_{\mathrm{hor}} \simeq 63 - 1 + 1 - 5 \pm 5 \tag{4.32}$$

$$= 58 \pm 5 \tag{4.33}$$

which is approximated to 55 ± 5 in their paper.

In the quartic case, however, they found that more precision is possible in estimating the number of e-foldings. This is because reheating in this potential obeys the same expansion law as in radiation domination. They obtained $N_{\mathrm{hor}}^{\mathrm{quartic}} \simeq 64$.

We also have to take into account the sort of scale on which we will evaluate our constraints, which is considerably smaller than the present Hubble horizon. The relevant scale for our observations is $k \simeq 0.01\,\mathrm{Mpc}^{-1}$ (Leach and Liddle, 2003b), and the present Hubble horizon is $a_0 H_0 \simeq 100h\,\mathrm{km\,s^{-1}\,Mpc^{-1}} \simeq h/3000\,\mathrm{Mpc}^{-1}$ (since the speed of light, $3\times 10^5\,\mathrm{km\,s^{-1}}$, is equal to 1 in our units), where $h \simeq 0.7$. Hence, from Liddle and Leach (2003, Eq. 6),

$$N(k) = -\ln\frac{k}{a_0 H_0} + N_{\mathrm{hor}} \tag{4.34}$$

$$\simeq -4 + N_{\mathrm{hor}}\,. \tag{4.35}$$

In the general case, this gives $N \simeq 54 \pm 5$ and in the quartic case this gives $N \simeq 60$.

Table 4.2: As Table 4.1, but only showing the quadratic potential, with different values of N, the number of e-foldings.

N	$n_{\rm low}-1$	$n_{\rm high}-1$	$R_{\rm low}$	$R_{\rm high}$
50	-0.040	-0.050	0.158	0.239
55	-0.036	-0.045	0.144	0.217
60	-0.033	-0.041	0.132	0.199

Table 4.3: As Table 4.1, but only showing the quartic potential, with different values of N, the number of e-foldings.

N	$n_{\rm low}-1$	$n_{\rm high}-1$	$R_{\rm low}$	$R_{\rm high}$
55	-0.053	-0.054	0.283	0.288
60	-0.049	-0.050	0.260	0.264

For simplicity, in all the following we therefore take $N = 55$ as a reasonable fiducial value, but will consider alternative values.

This approximation has very little effect on the high- and low-energy results as compared with $N = 54$, but there is the uncertainty of roughly 5 in either direction. The effect of this uncertainly on the perturbations is shown in Table 4.2, where values of N ranging from 50 to 60 are considered. For the quartic potential, with $N = 60$ and $\alpha = 4$, there is a difference compared with $N = 55$ of approximately 10%, as shown in Table 4.3. The results shown in both of these tables, along with the intermediate points, are shown in Fig. 5.3, where a comparison is made with observations (discussed in Chapter 5).

4.3 Varying brane tension with quadratic and quartic potentials

As well as the high- and low-energy limits, it will also be useful to obtain results for the intermediate regime, where the brane tension is comparable to the potential. Owing to the complexity of the equations, this has only proved possible for $\alpha = 2$ and 4, but this is sufficient, since it turns out that potentials of a higher exponent are excluded by the data (see Chapter 5).

The first step in finding n and R in terms of λ is to find the value of the scalar field at the end of inflation, $\phi_{\rm end}$, in terms of m and λ. From Eq. (4.3), η is a function of ϕ, m and λ, so this equation may be solved for ϕ with $\eta = 1$ (analytically, using Maple: see Appendix A) to obtain $\phi_{\rm end}(m, \lambda)$.

Second, it is necessary to find the value of the inflaton field when the observable perturbations were produced. Eq. (3.39) is an equation for N in terms of m, λ, $\phi_{\rm end}$ and ϕ_N, where ϕ_N is the value of the scalar field N e-foldings before the end of inflation. Having chosen $N = 55$, and

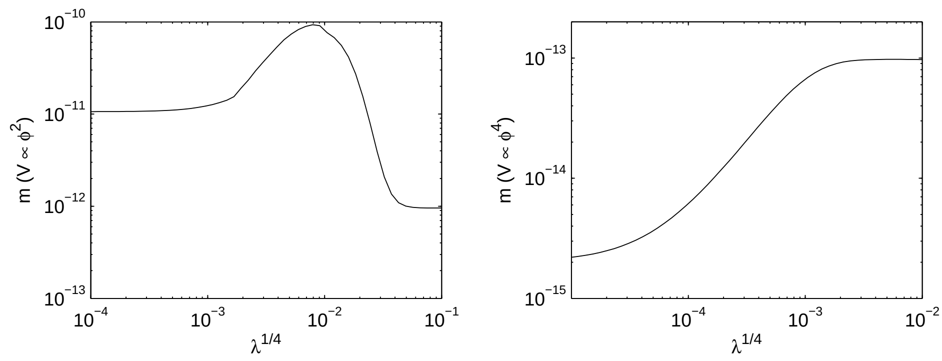

Figure 4.1: m against $\lambda^{1/4}$ for quadratic and quartic potentials, satisfying constraints from COBE. M_4 has been set to be equal to 1.

having found $\phi_{\rm end}(m,\lambda)$, this equation may be solved to get $\phi_{55}(m,\lambda)$. (It didn't prove possible to solve this equation for α higher than 4, which is why the complete results are only presented for two simple potentials.) It is this value of ϕ which is then used to calculate n and R in terms of m and λ. Again, this is done analytically, using Maple. The Maple program is reproduced in Appendix A.

The final step to obtain $n(\lambda)$ and $R(\lambda)$ is to impose the COBE normalization, in the form $A_{\rm S} = 2 \times 10^{-5}$ (Bunn et al., 1996, see Section 2.2.4). Eq. (3.41) for $A_{\rm S}^2$, which is evaluated at $\phi = \phi_{55}$, can be solved using this constraint to give $m(\lambda)$. This leaves λ as the only free parameter when determining the predicted perturbations.

For both the quadratic and quartic potentials, it did not prove possible to solve the equation for $A_{\rm S}$ using Maple, either analytically or numerically, so it was necessary to transfer the relevant formulae to Matlab, which has more sophisticated numerical solution algorithms. The expressions generated by the Maple program for $A_{\rm S}^2(m,\lambda)$, $n(m,\lambda)$ and $R(m,\lambda)$ were made into Matlab *m-files*. Matlab was then used to numerically solve $A_{\rm S}^2 = 4 \times 10^{-10}$ (the COBE normalization) for a range of different values of λ to give $m(\lambda)$. n and R were then evaluated at these values of λ to give the results. Appendix B contains the Matlab program for the quadratic potential.

Fig. 4.1 shows how m scales against λ for the quadratic and quartic potentials. The plot for the quadratic case may be contrasted with Fig. 1 of Maartens et al. (2000), which shows the same but with $\epsilon = 1$ marking the end of inflation.

Fig. 4.2 shows the predicted perturbations for the quadratic and quartic potentials as a function of the brane tension. They show that, for the quadratic potential, high energies (i.e., small values of λ) push the perturbations further from scale-invariance than in the low-energy limit, whereas, for the quartic potential, there is little difference between the high- and low-energy cases. In the intermediate regime, where the brane tension is comparable to the potential, the perturbations are driven further from scale-invariance than in either of the limits, for both potentials. In Fig. 5.1 these results are given on a plot of R against $n-1$ and compared with observations.

It has been claimed that braneworld effects ought to drive the perturbations towards the scale-

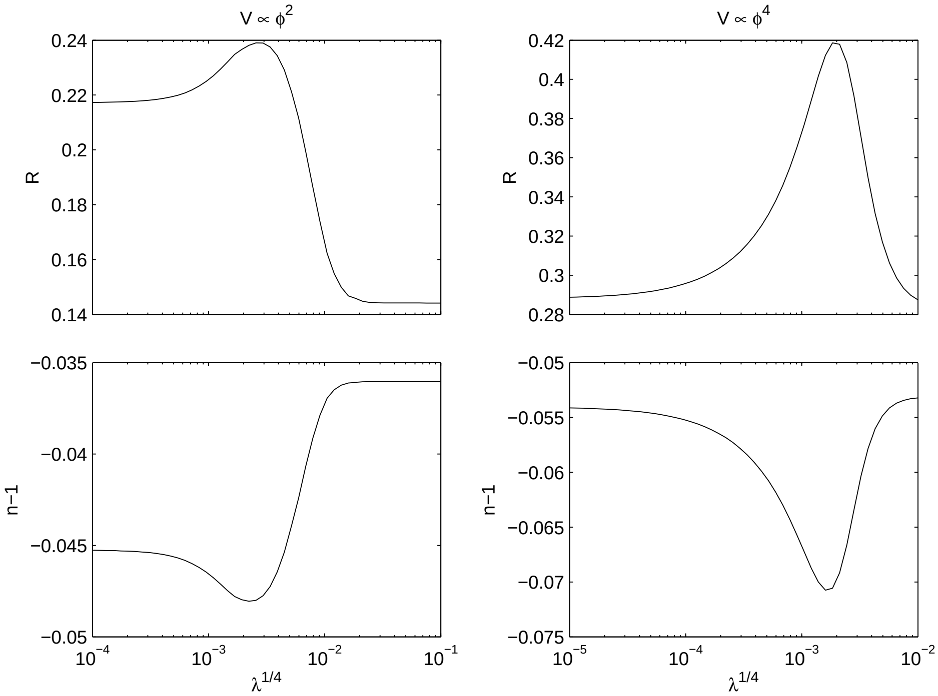

Figure 4.2: Theoretical predictions for $n-1$ and R against $\lambda^{1/4}$ for quadratic and quartic potentials. The models are all normalized to give the correct perturbation amplitude, and M_4 has been set to be equal to 1. (Figure as in Liddle and Smith, 2003.)

invariant Harrison–Zel'dovich spectrum for any potential (Maartens et al., 2000). For certain potentials, however, taking the details of the argument into account, I have found that this is not the case, with the perturbations at times being driven *further* from scale-invariance by braneworld corrections.

Chapter 5

Comparison with Observations

We are now in a position to test certain models of braneworld inflation against observations. We will compare the predictions of Chapter 4 with the observational constraints found by Leach and Liddle (2003b), as discussed in Section 2.2.4.

The cosmological data set used to obtain the constraints was taken from a variety of sources, from observations both of the cosmic microwave background and of galaxy clustering. These were the VSA (Very Small Array), CBI (Cosmic Background Imager), ACBAR (Arcminute Cosmology Bolometer Array Receiver) and WMAP (Wilkinson Microwave Anisotropy Probe) observations of the CMB, and the 2dF Galaxy Redshift Survey.

In Fig. 5.1, the predicted values shown in Fig. 4.2 are plotted on the n–R plane, with λ varying along the thick curves. Scale-invariance corresponds to the bottom-right of the figure. It may be seen that braneworld effects in the quadratic and quartic potentials move the perturbations further from scale-invariance, and that the effect is strongest in the intermediate regime, i.e., when the brane tension is comparable to the energy of the potential during inflation. The quartic potential may be seen to lie outside the three-sigma contour for all values of the brane tension, but this must be interpreted bearing in mind the uncertainty in the number of e-foldings (see Fig. 5.3). In fact, both of the curves should be somewhat blurred for this reason.

Fig. 5.2 shows how the high- and low-energy limits vary with the exponent of the potential. The points on the curves are calculated using Eqs. (4.17)–(4.20). Although the intermediate points have not been calculated for exponent higher than four, it is expected that they would form curves between the high- and low-energy points in much the same way as in the quadratic and quartic cases, as shown in Fig. 5.1. It may be seen that potentials of exponent higher than 4 lie comfortably outside the permitted range of values. In particular, the high-α limit in the high-energy case (steep inflation) is ruled out.

However, Liddle and Urena-Lopez (2003) have suggested an alternative method of reheating for the steep inflation model, which might bring that model back into the region allowed by observations. They note that the original steep inflation model, in which inflation is followed by a period

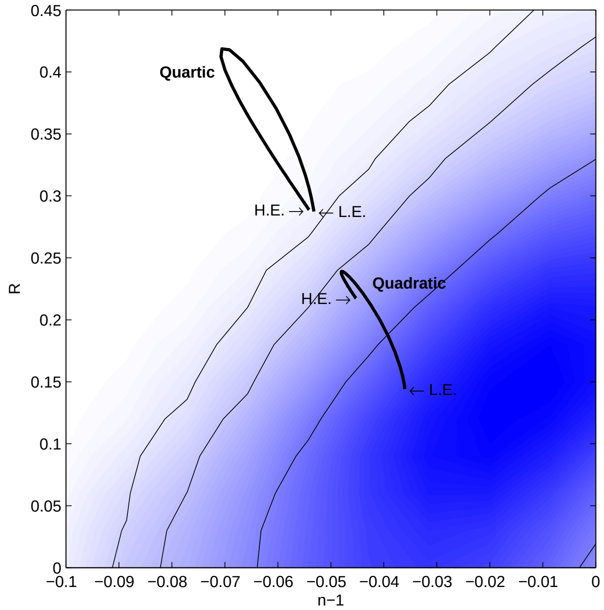

Figure 5.1: Theoretical predictions compared to observational constraints for the quadratic and quartic potentials, as a function of the brane tension λ, with $N = 55$. The low- and high-energy limits are shown. The observational contours are one-, two- and three-sigma confidence levels. (Figure as in Liddle and Smith, 2003, with observational constraints provided by Sam Leach.)

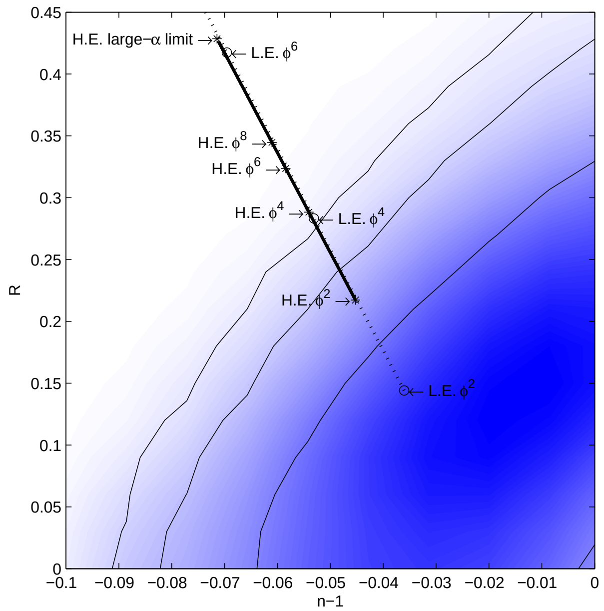

Figure 5.2: As Fig. 5.1, but now showing the low-energy limit (dotted line) and high-energy limit (solid line) as functions of α, for $\alpha \geq 2$. The locations corresponding to α being an even integer are highlighted, as is the large-α limit in the high-energy case. The number of e-foldings is fixed to be 55. (Figure as in Liddle and Smith, 2003.)

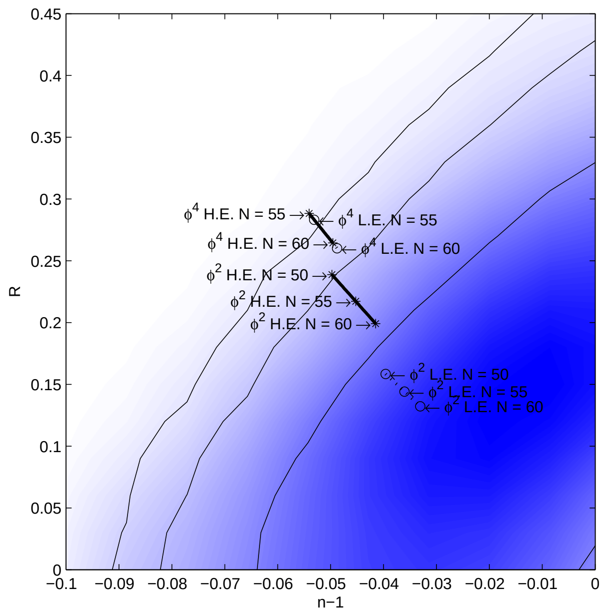

Figure 5.3: As Fig. 5.1, but showing the effect of varying the number of e-foldings on the high- and low-energy perturbations generated by the quadratic and quartic potentials. The points corresponding to $N = 50$, 55 and 60 in both limits are shown for the quadratic potential, and those corresponding to $N = 55$ and 60 are shown for the quartic potential.

of reheating involving gravitational particle production, does not agree with observations, since it produces an excessive amplitude of long-wavelength gravitational waves, and because the resulting perturbations are too far from scale-invariance. Instead, it is proposed that an alternative method of reheating, driven by a **curvaton field**, may help to solve these difficulties. The curvaton is a field which may have existed along with the inflaton field, and which may have been responsible for density perturbations.

Fig. 5.3 shows the effect of the uncertainty in the number of e-foldings on the predictions generated by the two potentials (as discussed in Section 4.2, and as shown in Tables 4.2 and 4.3). For the quartic potential, $N = 60$ corresponds to the more accurate estimate made possible by the unusual properties of the reheating era under this potential, and $N = 55$ is the approximation used in the other results. Taking the higher value for the number of e-foldings brings the low- and high-energy limits within the three-sigma contour, but still outside of the two-sigma contour. For the quadratic potential, the low-energy limit is comfortably within the observational constraints for all value $50 \leq N \leq 60$, but the high-energy limit comes under fairly strong pressure for a small number of e-foldings.

In the paper from the WMAP team detailing the implications for inflation (Peiris et al., 2003), the ϕ^4 potential is ruled out as lying at more than three-sigma away from the maximum likelihood point. However, the number of e-foldings they quote on p. 8 is $N = 50$—probably significantly too small. Barger et al. (2003) analysed the WMAP data and found that—for the WMAP data alone—

this three-sigma constraint only holds for $N < 45$. A larger number of e-foldings would place less pressure on this potential. The quartic potential does come under significant observational pressure, but not as strong pressure as has been claimed.

More recently, Kinney et al. (2003) have claimed that the quartic potential is ruled out at three-sigma for all values of N less than 66. However, Leach and Liddle (2003b) advise interpreting the results of this paper with caution (since they use the Monte-Carlo flow reconstruction technique, whereas Leach and Liddle use the Markov Chain Monte Carlo method) and in their Fig. 5 give constraints on the ϕ^4 potential in the low-energy limit which agree with my Fig. 5.3. (They take $N = 60$ as a maximum in that figure.)

For the low-energy limit, Leach and Liddle (2003b) found that the exponent, α, of the potential is constrained at different confidence levels to be

$$\begin{aligned} \alpha &< 4.3, \quad 3\sigma \\ \alpha &< 3.9, \quad 2\sigma \\ \alpha &< 2.8, \quad 1\sigma\,. \end{aligned} \tag{5.1}$$

The results in this chapter show that (assuming $N = 55$) the constraint on the exponent is tighter in the high-energy limit, at one-, two- or three-sigma, than in the low-energy limit. This is clear from Fig. 5.2. The exponent in this case is bounded above by some value between 2 and 4, depending on the accuracy required. In the intermediate regime, where the brane tension is comparable to the inflationary energy scale, it has been found that the quadratic and quartic potentials come under stronger observational pressure that in either of the limits, and the quartic potential is strongly ruled out in the majority of this region.

Chapter 6

Conclusions

6.1 Summary of results

We have seen that the perturbations generated by a given potential in the Randall–Sundrum type II braneworld are dependent on the brane tension. In particular, predictions for the scalar spectral index and relative strength of tensor perturbations have been found for monomial potentials in the high- and low-energy limits, and for the quadratic and quartic potentials for any value of the brane tension. These results have been compared with observations, including WMAP, enabling constraints to be placed on models of braneworld inflation.

It may be expected, given the additional friction in the Friedmann equation, that braneworld effects would drive the perturbations nearer to scale-invariance. I have found that this is not necessarily the case, particularly when the inflationary energy scale is comparable to the brane tension. I have also found that the constraint on the exponent of monomial potentials is strengthened by brane effects, and also that the original steep inflation model is excluded by the data.

Does this make it possible to fix the value of the brane tension? If we knew the exact form of the potential, we would (in principle) be able to determine the brane tension from observations. But the potential may take essentially any form, and we do not know the form that it takes. Indeed, Liddle and Taylor (2002) have shown that it is not possible to reconstruct the inflaton potential from the initial perturbations. That is, given the spectra of perturbations, for *any* value of the brane tension, there exists a potential which will produce those perturbations. This follows from the consistency equation, (3.56), which reduces the number of independent inflationary parameters and hence introduces some degeneracy.

This degeneracy is found when considering the initial perturbations only. However, there may be effects caused by the braneworld on the subsequent evolution of these perturbations. The details of this are presently unknown, but this may offer some hope of reconstructing the potential from observations of large-scale structure, for example.

The Randall–Sundrum braneworld would also affect gravity on very small scales. Experiments have tested gravity down to 0.1 mm (Long et al., 2003), and no departure from Newtonian gravity has been detected. These experiments would place a constraint on the brane tension, which will become more strict as experiments probe smaller length scales.

6.2 Alternative models

It would be a mistake to give the impression that the Randall–Sundrum braneworld investigated here is the only viable braneworld model, or even that inflation is universally accepted. In reality the situation is more complicated.

In the Randall–Sundrum type II braneworld, we have taken the bulk to possess a cosmological constant only. This is not a necessary restriction on braneworld models. For example, there may be a scalar field in the bulk (see, for example, Himemoto and Sasaki, 2001). This would add to the complexity of the model, but it is conceivable that such a field may have left traces in the CMB or large-scale structure (Rhodes et al., 2003).

Other models exist (for a review, see Brax and van de Bruck, 2003), in which there may be more than one brane, or in which the branes may be moving, or even colliding.

A radically different model featuring colliding branes is the **Ekpyrotic** model of Khoury et al. (2001). This has the Big Bang caused by the collision of two branes. In an extension to this, the **Cyclic Universe**, these branes collide repeatedly, over a very long period of time, causing a succession of many Big Bangs. It is claimed that this model would solve the classic problems of the (non-inflationary) Big Bang cosmology without invoking a period of inflationary expansion. It also seeks to explain the cosmological constant problem of inflation, of why the current cosmological constant is so small, by incorporating the cosmological constant as a fundamental component of the theory.

Brandenberger (2001) has outlined some of the problems of the inflationary paradigm. For example, he notes that scales of cosmological interest today would correspond to lengths shorter than the Planck length at the start of inflation. However, physical properties on such length scales are not well understood at present. There is therefore a need to develop a proper string cosmology, using more detailed predictions from string/M-theory. This will have to follow from a more complete understanding of those theories, which is likely to take many years.

6.3 Future prospects

Despite the vast number of alternative models, the Randall–Sundrum type II braneworld is still a useful model to consider. It has a simplicity which enables details to be investigated—and simplicity is a good thing in a theoretical physics. It also has the obvious advantage that it gives 'ordinary' gravity at low energies.

Forthcoming observations will help to constrain braneworld inflationary models, even if they do not completely remove the degeneracies. In terms of the CMB, the WMAP mission is yet to be completed, and more accurate data are to be released in the not-too-distant future. The Planck satellite[1] is intended to measure the anisotropies in the microwave background to an even higher degree of accuracy. This is scheduled to be launched in 2007. Measurements of the polarization of the CMB (e.g., by Planck and by MAXIPOL[2]) may also help constrain the inflationary parameters, particularly those relating to primordial gravitational waves. It is also hoped that gravitational waves may be detected directly in the near future, for example by LISA[3] (Laser Interferometer Space Antenna).

[1] `http://astro.estec.esa.nl/Planck/`

[2] `http://groups.physics.umn.edu/cosmology/maxipol/`

[3] `http://lisa.jpl.nasa.gov/`

Acknowledgements

First of all, I would like to thank my supervisor, Professor Andrew Liddle, for excellent supervisions, a clear sense of direction for the project and helpful advice throughout the year.

Thanks also to Sam Leach, who provided data from the observational constraints published in Leach and Liddle (2003a) in a form suitable for comparison with my results (see Chapter 5).

Many thanks to the other people in the department for making my year so enjoyable. Thanks to Alex, Laura and Mike (the MSc astronomy students), for being such great people, and for giving me a reason to eat chocolate by the endless amusement you derive from the most feeble attempts at humour (not mine, I hasten to add!). Also to the rest of you, for making the department such a cheerful place, including: Natalie, Payam, Neil, Michael, James, Fernando, ...

Outside university, there are many people whom I would like to thank for their support through the year. I will mention a few. Many thanks, of course, to my parents, for keeping me alive. Thanks to Indarto, Mark, Jason, JJ and Dave for putting up with me as a housemate (even if only one of you could cope with me for the whole year!). Thanks to Phil and Chris for your friendship, encouragement and cake. And thanks to all my friends at Calvary Evangelical Church for your prayerful support and for making me so welcome—particularly to the Wells family, for your limitless hospitality and for making me feel very much a part of the family. Philip: if Hollywood do turn this project into a movie, *Braneworld*, (which, let's face it, they'd be foolish not to) I'll make sure you are acknowledged for the idea!

Appendix A

Maple V Program

This program generates (very long) expressions for A_{S}^2, n and R, which are then used by the Matlab program (Appendix B) to generate the plots shown in the main body of the text.

With V = alpha*phi^(2*beta), given beta, solve eta=1 and N=55 to obtain A_S^2, n and R in terms of lambda and alpha

Find phi_55 in terms of lambda and alpha

NB different from thesis text, where V = m*phi^alpha

```
> restart:
```

Make assumptions about alpha, or it won't solve for phififtysq

```
> assume(alpha,real):
> assume(alpha>0):
> assume(phisq,real):
> assume(phisq>0):
> assume(lambda,real):
> assume(lambda>0):
```

Execute these lines to check that chosen solutions for phififtysq and phiendsq are positive

```
> alpha:=1e-10:
  lambda:=1e-12:
```

Choose value of beta: V=alpha*phi^(2*beta)

```
> beta:=2:
```

Definitions

```
> mfour:=1:
  V:=alpha*phisq^beta:
  Vprimesq:=4*alpha^2*beta^2*phisq^(2*beta-1):
> Vdoubleprime:=2*alpha*beta*(2*beta-1)*phisq^(beta-1):
> Hsq:=8*Pi*V*(1+V/(2*lambda))/(3*mfour^2):
> epsilon:=(mfour^2/(16*Pi))*(Vprimesq/V^2)*(2*lambda*(2*lamb
  da+2*V)/(2*lambda+V)^2):
> eta:=(mfour^2/(8*Pi))*(Vdoubleprime/V)*(2*lambda/(2*lambda+
  V)):
```

NB d(phisq) rather than d(phi). Also phififtysq really phi_55^2

```
> N:=-8*Pi/mfour^2*int(V/sqrt(Vprimesq)*(1+V/2/lambda)/2/sqrt
  (phisq),phisq=phififtysq..phiendsq):
```

Solve for phi at end of inflation (change suffix to get a positive root) (beta=1: [1]; beta=2: [1])

```
> solve(1 = eta,phisq);
> phiendsq := solve(1 = eta,phisq)[1];
```

Solve for phi at 55 e-folding before end of inflation (change suffix for +ve root) (beta=1: [1]; beta=2: [1])

```
> phififtysqsol:=solve(55=N,phififtysq);
> phisq:=phififtysqsol[1];
```

Display expressions for n, A_S^2 and R, for numerical solution and evaluation using Matlab

Definitions again, with phi=phi_55, which is a function of lambda and alpha

```
> V:=alpha*phisq^beta:
  Vprimesq:=4*alpha^2*beta^2*phisq^(2*beta-1):
> Vdoubleprime:=2*alpha*beta*(2*beta-1)*phisq^(beta-1):
> epsilon:=(mfour^2/(16*Pi))*(Vprimesq/V^2)*(2*lambda*(2*lamb
  da+2*V)/(2*lambda+V)^2):
> eta:=(mfour^2/(8*Pi))*(Vdoubleprime/V)*(2*lambda/(2*lambda+
  V)):
```

Displays n in terms of lambda and alpha

```
> n := 1-6*epsilon+2*eta;
```

More definitions

```
> Hsq:=8*Pi*V*(1+V/(2*lambda))/(3*mfour^2):
  mu:=sqrt(4*Pi/3*lambda)/mfour:
  x:=sqrt(Hsq)/mu:
  Fsq:=1/(sqrt(1+x^2)-x^2*ln(1/x+sqrt(1+1/x^2))):
  ATsq:=4*Hsq*Fsq/(25*Pi*mfour^2):
```

Displays A_S^2 in terms of lambda and alpha

```
> ASsq:=512*Pi*V^3*(1+V/(2*lambda))^3/(75*mfour^6*Vprimesq);
```

Displays R in terms of lambda and alpha

```
> R:=16*ATsq/ASsq;
```

Appendix B

Matlab 6 Program

This program generates the plots for the quadratic potential. (The program for the quartic potential is almost identical.) `assq2`, `ncalc2` and `rcalc2` are m-files which contain the expressions generated by the Maple program (Appendix A), with `beta=1`. `assq2` takes `lambda` and `alpha` and returns $A_S^2 - 4 \times 10^{-10}$, so `assq2=0` corresponds to the COBE normalization.

```
% With N=55 and R=16(A_T^2/A_S^2), V=alpha phi^2
% Generates plots for alpha, n and R in terms of lambda
% and plot of n against R
global lambda
lfourth=logspace(-4,-1);
eps=1e-30;
for i=1:50
    lambda=lfourth(i)^4;
    alpha(i)=fsolve(@assq2,1e-10,optimset('TolX',1e-30,...
       'TolFun',1e-50));
end
alpha(50)
figure
loglog(lfourth,alpha)
title('\alpha for V = \alpha\phi^2')
xlabel('\lambda^{1/4}')
ylabel('\alpha')

for i=1:50
    n2(i)=ncalc2(alpha(i),lfourth(i)^4);
end
```

```
n2(50)
figure
semilogx(lfourth,n2)
title('n for V = \alpha\phi^2')
xlabel('\lambda^{1/4}')
ylabel('n')

for i=1:50
    r2(i)=rcalc2(alpha(i),lfourth(i)^4);
end
r2(50)
figure
semilogx(lfourth,r2)
title('r for V = \alpha\phi^2')
xlabel('\lambda^{1/4}')
ylabel('r')

figure
plot(n2-1,r2)
title('n and r for V = \alpha\phi^2')
xlabel('n-1')
ylabel('r')
```

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